

A Critique of Euler's Submission that 'Squares of Numbers are Always Positive'

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Abstract

The work critiques Euler's submission that the squares of (negative or positive) numbers are always positive, that is, for any number x , $(-x)^2 = x^2 \geq 0$. The work looks at many equations in which the case is not that $(-x)^2 = x^2 \geq 0$ and concludes that $(-x)^2 = x^2 \geq 0$ is not always true except the number x is zero. The work recommends that mathematicians and scientists should admit existence of a Platonic number $i_* = (-1)^2$ and also admit that given a number $x \neq 0$, $(-x)^2 \neq x^2$.

Text Words

Binomial Functions, Complex Number, Negative Number, Square, Real Number

1.0 Background to the Study

This study emanated from attempts to find z that satisfies the complex binomial equation $1 - \sqrt{1-4z} = 2$. The equation is the same as $1 + \sqrt{1-4z} = 0$. It turns out that based on Euler's submission no number in the entire set of complex numbers can satisfy that equation! Euler (in [2]) states that "the squares of numbers, negative as well as positive, are always positive". By Euler's submission, the answer to that problem would be $z = 0$ but putting this value back into the equation gives $1 = 0$, which is a contradiction. Indeed mathematicians have warned that for a field it is necessary that $1 \neq 0$.

The implication of Euler's submission is that for any complex (or real) number x , $(-x)^2 = x^2 \geq 0$. For instance, $(-2)^2 = 4 = 2^2$, which is founded on the erroneous submission that $(-1)^2 = 1$. To show that $(-x)^2 = x^2$ is not always true, is the motivation of this study. For often, when that result $((-x)^2 = x^2)$ is put back into an original problem it fails the test of being true. Thus a Platonic mathematical object must be found to satisfy that equation. For, enlightenment as led by Immanuel Kant demands for reason(s) for any phenomenon to be accepted universally as truth, which requires vigorous proof; that is, a logical reasoning as evidence which takes the original mathematical statement or problem (or system) to its tautology ($0 = 0$). Thus any proof for truth of a system or an equation that is not tautological is not vigorous and indeed false.

1.1 Definition of Terms

Important concepts of the study are defined as used in the work. They include:

1.2 Binomial Function

The binomial function is of the form $f(x) = (t + x)^n = \sum_{k=0}^n \binom{n}{k} t^n x^{n-k}$ for n real and t, x a real (or complex) variable with $\binom{n}{k} = \frac{n!}{(n-k)!k!} : 0! = 1! = 1$ [3].

1.3 Complex Number

A complex number is a number of the form $m + ni : i^2 = -1$, m, n are real numbers [1].

1.4 Negative Number

A negative number x is one whose value is less than zero. That is, $x < 0$ [1].

1.5 Square

By square of x is meant a number x multiplying itself twice. That is, $x^2 = x.x$.

1.6 Real Number

A real number is any of integers, rational or irrational numbers [1].

1.7 Aim of the Study

The aim of the study is to show that the square of a number is not equal to the square of its negative unless such a number is zero. The motivation of the study is also to apply the results of the study to doing more mathematics.

2.0 Method and Materials

Squares and square roots of numbers are taken for assessment of expressions. Binomial expressions of half-fractional powers are used as materials.

3.0 Results and Remarks

Theorem:

The expression $(-x)^2 = x^2$ for some number (x), is not always true except for $x = 0$. That is,

$$(-x)^2 \neq x^2 : x \neq 0 \quad (1)$$

Proof:

Let x be any expression or number (complex or real). Suppose that $(-x)^2 = x^2$. Then raising both sides to power half,

$$\begin{aligned} ((-x)^2)^{\frac{1}{2}} &= (x^2)^{\frac{1}{2}} \\ \Rightarrow -x &= x, \end{aligned}$$

which is a contradiction except for $x = 0$. Thus $(-x)^2 \neq x^2$ except for $x = 0$. It is thus sacrosanct that

$$(-x)^2 = (-x)^2 \quad (2),$$

since $(-x)^2 = (-x)^2 \Rightarrow -x = -x \Rightarrow 0 = 0$, the expected tautology for vigorous proof.

3.1 Remark 1

Let $x = 1$. The immediate consequence of this result is that $(-1)^2 \neq 1$.

Proof:

From orthodoxy, $(-1)^2 = 1$. Suppose that this is true, then taking the half power of both sides, $-1 = 1$, which is a contradiction. Thus $(-1)^2 \neq 1$, that is a sacrosanct:

$$(-1)^2 = (-1)^2 \quad (3)$$

since raising both sides to power half gives the tautology $-1 = -1 \Rightarrow 0 = 0$.

3.2 Remark 2

Thus $(-1)^2$ must be recognized as a special number, a Platonic mathematical object, of great significance in mathematics and science. Denoting $(-1)^2 = i_*$, then

$$i_*^{\frac{1}{2}} = -1, \text{ or } \sqrt{i_*} = -1 \quad (4).$$

This implies that there is no number m such that $\sqrt{m} = -1$ except $m = (-1)^2 \neq 1$. For if we admit $(-1)^2 = 1$ then $\sqrt{m} = -1 \Rightarrow m = (-1)^2 = 1$ which when put back into $\sqrt{m} = -1$ it yields $\sqrt{m} = \sqrt{1} = 1 \neq -1$.

3.3 Remark 3

The result that $(-x)^2 \neq x^2$ is useful in further doing mathematics and science. Suppose, for instance, that the problem is to find z in a binomial expression with half as power, that is,

$$\sqrt{1+(-z)^2} = -k \quad (5).$$

Now, by Euler, $(-z)^2 = z^2$, and

$$\begin{aligned} (-k)^2 &= k^2 \\ \Rightarrow \sqrt{1+(-z)^2} &= \sqrt{1+z^2} = -k \\ \Rightarrow 1+z^2 &= (-k)^2 = k^2 \\ \Rightarrow z &= \sqrt{k^2-1}. \end{aligned}$$

Putting back $z = \sqrt{k^2-1}$ into (5), $\Rightarrow \sqrt{1+(-z)^2} = \sqrt{1+z^2} = \sqrt{1+(\sqrt{k^2-1})^2} = \sqrt{k^2} = k \neq -k$.

This result is disastrous! The disaster can, however, be rectified if we preserve $(-z)^2 = (-z)^2$: Thus

$$\begin{aligned}\sqrt{1+(-z)^2} &= -k. \\ \Rightarrow 1+(-z)^2 &= (-k)^2 \\ \Rightarrow (-z)^2 &= (-k)^2 - 1 \\ \Rightarrow -z &= [(-k)^2 - 1]^{\frac{1}{2}} \\ \Rightarrow z &= -[(-k)^2 - 1]^{\frac{1}{2}}, \text{ or } : z = -\sqrt{[(-k)^2 - 1]}.\end{aligned}$$

Putting back $z = -\sqrt{(-k)^2 - 1}$ into (8), we then have

$$\sqrt{1+(-z)^2} = \sqrt{1+\left(-\sqrt{(-k)^2 - 1}\right)^2} = \sqrt{1+\left(\sqrt{(-k)^2 - 1}\right)^2} = \sqrt{1+(-k)^2 - 1} = -k,$$

as expected. Thus $(-z)^2 \neq z^2$, and $(-k)^2 \neq k^2$.

3.5 Remark 4

Suppose that $k = 2$ for finding z in Remark 3:

$$\sqrt{1+(-z)^2} = -k = -2 \quad (6).$$

Now, taking the orthodox $(-z)^2 = z^2, (-2)^2 = 2^2 = 4$

$$\begin{aligned}\Rightarrow \sqrt{1+(-z)^2} &= \sqrt{1+z^2} = -2 \\ \Rightarrow 1+z^2 &= (-2)^2 = 2^2 = 4 \\ \Rightarrow z &= \sqrt{4-1} = \sqrt{3}.\end{aligned}$$

$z = \sqrt{3} \Rightarrow \sqrt{1+(-z)^2} = \sqrt{1+z^2} = \sqrt{1+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2 \neq -2$, an ugly situation.

The truth rears if we preserve $(-z)^2 = (-z)^2$ for any number z . Thus

$$\begin{aligned}\sqrt{1+(-z)^2} &= -2. \\ \Rightarrow 1+(-z)^2 &= (-2)^2 \\ \Rightarrow (-z)^2 &= (-2)^2 - 1 \\ \Rightarrow -z &= [(-2)^2 - 1]^{\frac{1}{2}} \\ \Rightarrow z &= -[(-2)^2 - 1]^{\frac{1}{2}}, \text{ or } : -\sqrt{[(-2)^2 - 1]}.\end{aligned}$$

Putting back $z = -\sqrt{(-2)^2 - 1}$ into (9), we then have

$$\sqrt{1+(-z)^2} = \sqrt{1+\left(-\sqrt{(-2)^2 - 1}\right)^2} = \sqrt{1+\left(\sqrt{(-2)^2 - 1}\right)^2} = \sqrt{1+(-2)^2 - 1} = -2,$$

as expected.

3.6 Remark 5

There are practical applications of the results of the study. The following are such applications.

3.6.1 Find z in $\sqrt{-z} = -x$

By orthodoxy, $\sqrt{-z} = -x \Rightarrow -z = (-x)^2 = x^2 \Rightarrow z = -x^2$ which makes the original expression to fail to balance. That is: $\sqrt{-z} = \sqrt{-x^2} = \sqrt{x^2} = x \neq -x$.

Thus to resolve the problem, we preserve $(-x)^2$ as $(-x)^2$. That is,

$\sqrt{-z} = -x \Rightarrow -z = (-x)^2 \Rightarrow z = -(-x)^2$. Putting back this result into the original expression,

$\sqrt{-z} = \sqrt{-(-x)^2} = \sqrt{(-x)^2} = -x$ as desired.

3.6.2 Find x in $\sqrt{1+x} = -3$.

By Euler's submission: $\sqrt{1+x} = -3 \Rightarrow 1+x = (-3)^2 = 9 \Rightarrow x = 8$ which if put into the original problem gives $\sqrt{1+x} = \sqrt{1+8} = \sqrt{9} = 3 \neq -3$, a woeful failure at resolving the problem!

However preserving $(-x)^2 = (-x)^2$, we have $\sqrt{1+x} = -3 \Rightarrow 1+x = (-3)^2 \Rightarrow x = (-3)^2 - 1$ which put back into the original problem gives $\sqrt{1+x} = \sqrt{1+(-3)^2 - 1} = \sqrt{(-3)^2} = -3$ as expected.

3.7 Remark 6

Thus $(-i)^2 \neq i^2$ (7).

Proof:

Suppose that $(-i)^2 = i^2$, then taking half-power of both sides, $-i = i$ which is a contradiction. Thus $(-i)^2 \neq i^2$. We must thus preserve $(-i)^2$ as $(-i)^2$.

3.8 Remark 7

The norms of spaces too need to be carefully calculated, consequently. For instance, the norm $|-1-i| = \sqrt{(-1)^2 + (-1)^2} \neq \sqrt{2}$ as orthodoxly held using Euler's submission.

Proof:

Suppose that $|-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Then taking half-power of both sides

$$(-1)^2 + (-1)^2 = 2$$

$$\Rightarrow 2(-1)^2 = 2$$

$$\Rightarrow (-1)^2 = 1$$

$$\Rightarrow -1 = 1,$$

a contradiction.

Thus, by preserving $(-1)^2 = (-1)^2$, the norm

$$|-1-i| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2(-1)^2} = -\sqrt{2}.$$

This result can be verified by:

$$\begin{aligned} \sqrt{(-1)^2 + (-1)^2} &= -\sqrt{2} \\ \Rightarrow \left(\sqrt{(-1)^2 + (-1)^2}\right)^2 &= (-\sqrt{2})^2 = \left(\sqrt{2(-1)^2}\right)^2 \\ \Rightarrow (-1)^2 + (-1)^2 &= 2(-1)^2 \\ \Rightarrow 2(-1)^2 &= 2(-1)^2 \\ \Rightarrow 1 &= 1 \\ \Rightarrow 0 &= 0. \end{aligned}$$

3.9 Remark 8

An implication of this discovery is that norms of complex numbers could be negative real numbers. This makes it imperative the redefinition of complex metric spaces as being of either positive or negative real values, an example being $|-1-i| = -\sqrt{2}$.

3.10 Remark 9

Now the imaginary number $i : i^2 = -1$. Denote $i_* = (-1)^2$. Then $i_* = (i^2)^2 = i^4$. Thus a Platonic mathematical object i_* could be said to have been discovered and denoted $i_* = (-1)^2 = i^4$.

With $i_* = (-1)^2$ the problem that motivated this study, that is, $1 - \sqrt{1-4z} = 2$ can be resolved. That is, $z = \frac{1 - (-1)^2}{4} = \frac{1 - i_*}{4}$ satisfies $1 - \sqrt{1-4z} = 2$.

3.10 Remark 10

Thus admitting and preserving $i_* = (-1)^2 = i^4$ enables solving such problems as: Given $x^2 + x + c = 0$ find c which satisfies the part of the quadratic formula $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = 0$.

Solution using Euler's $(-x)^2 = x^2 \geq 0$:

$$a = b = 1$$

$$\Rightarrow -1 - \sqrt{1 - 4c} = 0$$

$$\Rightarrow -1 = \sqrt{1 - 4c}$$

$$\Rightarrow 1 - 4c = (-1)^2 = 1$$

$$\Rightarrow c = 0.$$

The value $c = 0$ in

$$-1 - \sqrt{1 - 4c} = -1 - \sqrt{1 - 4(0)} = -1 - 1 = -2 \neq 0.$$

This makes nonsense Euler's submission that squares of numbers whether positive or negative are always positive.

Indeed $c = 0$ in

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} = 0$$

$$\Rightarrow -b - \sqrt{b^2 - 4a(0)} = 0$$

$$\Rightarrow -b - \sqrt{b^2} = 0$$

$$\Rightarrow -b - b = 0$$

$$\Rightarrow -2b = 0; \text{ while } : b = 1$$

$$\Rightarrow -2 = 0$$

$$\Rightarrow 1 = 0.$$

This is contradictory.

Solution preserving

$$(-x)^2 = (-x)^2 :$$

$$a = b = 1$$

$$\Rightarrow -1 - \sqrt{1 - 4c} = 0$$

$$\Rightarrow -1 = \sqrt{1 - 4c}$$

$$\Rightarrow 1 - 4c = (-1)^2$$

$$\Rightarrow c = \frac{1 - (-1)^2}{4}.$$

The value $c = \frac{1 - (-1)^2}{4}$ in

$$-1 - \sqrt{1 - 4c}$$

$$= -1 - \sqrt{1 - 4\left(\frac{1 - (-1)^2}{4}\right)}$$

$$= -1 + 1$$

$$= 0$$

as expected.

References

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