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Acceleration of A Generalized Alternating Series

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Abstract:

In this manuscript, a novel approach is proposed for the approximation of alternating series $J^*(a,b)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(an-b)}$, where $a,b\in R$ with $a\neq 0$ by using a correction function. This method entails the construction of a correction function that captures the difference between the actual series and its intended approximation. By the incorporation of this correction function into the approximation process, an effort is made to enhance the accuracy and convergence of the alternating series.

Key Words: Alternating Series, Remainder term, Error term, Rate of convergence, Convergent series.

Introduction:

The approximation of an alternating series can be done using the remainder term [10] of the series. The absolute value of the remainder term is the correction function. The introduction of a correction function certainly improves the sum of the series and gives a better approximation for it. We can deduce generalized convergent series $J^*(a,b)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(an-b)}$, where $a,b\in R$ with $a\neq 0$ [14] using a correction function and the corresponding error functions. The new series so extracted increases the rate of convergence of the series.

Theorem 1.1:

The correction function for the series $J^*(a,b)$ is $G_n=\frac{1}{2an+a-2b}$ and the corresponding correction error function is $|E_n|=\left|\frac{a^2}{a^3\left(3+11n+12n^2+4n^3\right)-a^2b\left(11+24n+12n^2\right)+12ab^2\left(1+n\right)-4b^3}\right|$

Proof:

If G_n is the correction function after n terms of the given series , then we have $G_n+G_{n+1}=\frac{1}{an+a-b}$

The error function is $E_n=G_n+G_{n+1}-\frac{1}{an+a-b}$

We may select G_n in such way the $|E_n|$ is a minimum.

For fixed n and $r\in R$, consider $G_n(r)=\frac{1}{2an+2a-2b-r}$

So, the error function $E_n\left(r\right)=G_n\left(r\right)+G_{n+1}\left(r\right)=\frac{1}{an+a-b}$ is a rational function of r

Here, $E_n(r)=\dfrac{N_n(r)}{D_n(r)}$

$D_n(r)=(2an+a-2b)(2an+3a-2b)(an+a-b)\approx 4a^3n^3$, which is maximum for large values of n.

$$N_n(r)=\begin{cases}(r-a)(2an+2a-2b-r)+ar & ,r\neq a\\ a^2 & ,r=a\end{cases}$$

Therefore $|N_n(r)|$ minimum for $r=a$.

Hence, $|E_n(r)|$ minimum for $r=a$.

So, The correction function of the series $J^*(a,b)$ is $G_n=\dfrac{1}{2an+a-2b}$ and an absolute value of the corresponding error function is

$$|E_n|=\left|\frac{a^2}{a^3\left(3+11n+12n^2+4n^3\right)-a^2b\left(11+24n+12n^2\right)+12ab^2(1+n)-4b^3}\right|$$

Remark 1.2

The correction function $G_n<|a_{n+1}|\ ,\forall\ n\in N$, that is $G_n<\dfrac{1}{an+a-b}$

Remark 1.2

For $a,b\in N$ with $0<b<a$ we have, $\dfrac{1}{2an+2a-2b}<\dfrac{1}{2an+a-2b}<\dfrac{1}{2an-2b}\ ,\forall\ n\in N$ that implies $\dfrac{1}{2}|t_{n+1}|<G_n<\dfrac{1}{2}|t_n|$, where t_n denotes n^{th} term of the series.

Examples 1:

The correction function and corresponding error function of some series tabulated follows:

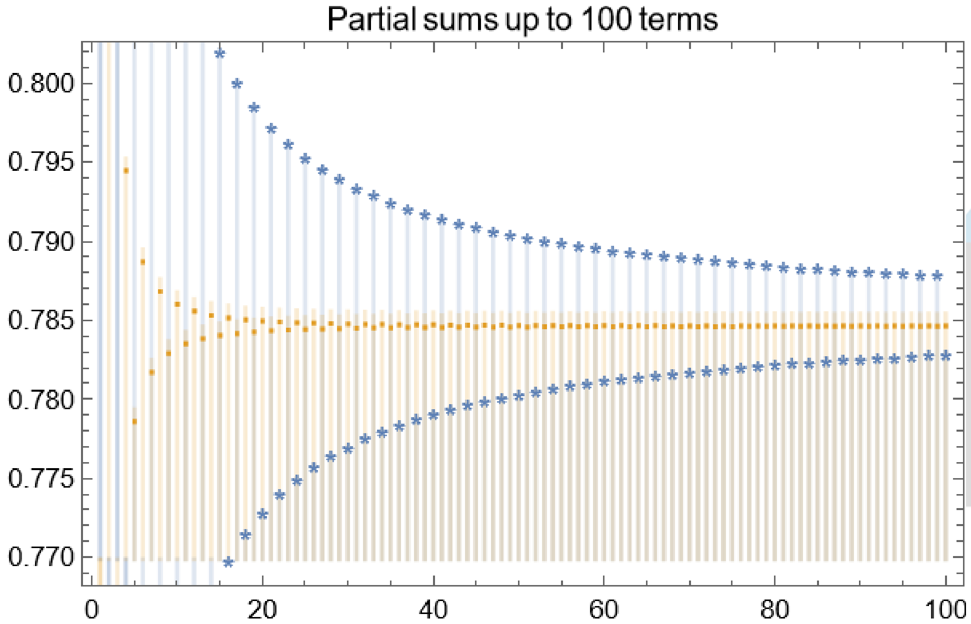
Sr.	**Alternating Series**	**Correction term - G_n**	**Error function $\|E_n\|$**
1	$J^*(2,1)$	$\dfrac{1}{4n}$	$\dfrac{1}{4n+12n^2+8n^3}$
2	$J^*(3,2)$	$\dfrac{1}{-1+6n}$	$\dfrac{9}{-5+9n+108n^2+108n^3}$
3	$J^*(5,3)$	$\dfrac{1}{-1+10n}$	$\dfrac{25}{-18+115n+600n^2+500n^3}$
4	$J^*(1,\frac{3}{4})$	$\dfrac{2}{-1+4n}$	$\dfrac{16}{-3-4n+48n^2+64n^3}$
5	$J^*(\sqrt{3},1)$	$\dfrac{1}{-2+\sqrt{3}+2\sqrt{3}n}$	$\dfrac{3}{-37+21\sqrt{3}+3n\left(3\left(-8+5\sqrt{3}\right)+4n\left(-3+3\sqrt{3}+\sqrt{3}n\right)\right)}$
6	$J^*(6,2)$	$\dfrac{1}{2+12n}$	$\dfrac{9}{28+234n+432n^2+216n^3}$

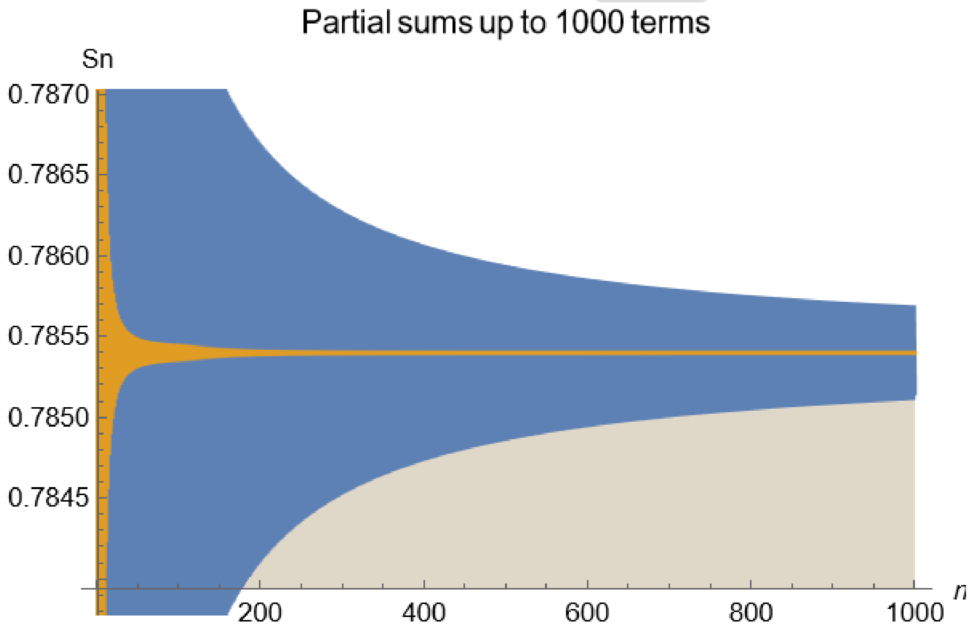
Application:

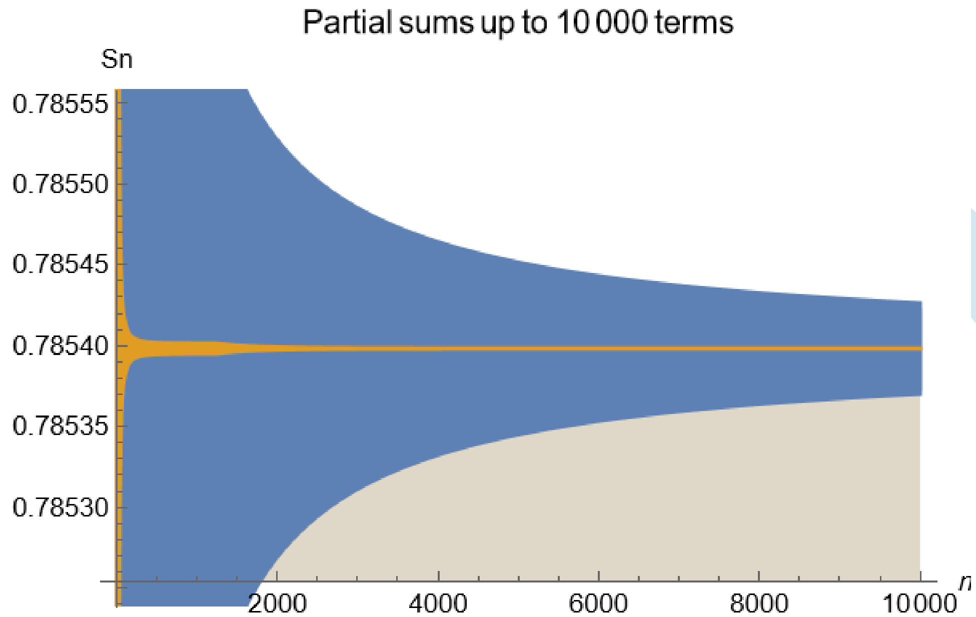
The series $J^{*}(2,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}$ is convergent and converges to $\frac{\pi}{4}=0.785398163398$. If S_n denotes the sequence of partial sums then by using correction function G_n, obtained approximation is given below:

Number of terms	S_n	$S_n+(-1)^n G_n$
10	**0.7**604599047323506	**0.78**67756942060348
100	**0.78**28982564911668	**0.7854**108193052372

In following Fig[1], Fig[2] and Fig[2], we plot sequence of partial sums S_n and $S_n+(-1)^n G_n$ of the series $J^{*}(2,1)$ up to 100, 1000 and 1000 terms respectively. In following figures we can see that by using correction function G_n series converges exponentially more rapidly than original series $J^{*}(2,1)$.







Conclusion:

In summary, leveraging generating functions for approximating alternating series proves to be a robust approach, effectively addressing complexities inherent in such series. This technique enhances accuracy and convergence analysis, offering a promising avenue for advancing series approximation methodologies.

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