

Review of Graph Theory And It's Application

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Abstract

Graph theory is a fundamental branch of discrete mathematics that studies the proper-ties and applications of graphs—mathematical structures composed of vertices (nodes) and edges (connections). The role of graph theory is pivotal in solving problems related to connectivity, optimization, and network structures. It has diverse applications in algorithm design, artificial intelligence, data science, transportation systems, and social network analysis. This abstract pro-vides an overview of the significance of graph theory, highlighting its key contributions to various disciplines and its potential for future advancements in computational and theoretical research. Graph theory plays a crucial role in various fields by providing a mathematical framework for analyzing relationships, structures, and networks. Its abstract nature allows for applications in computer science, biology, social sciences, and engineering. The world of system of systems engineer-ing (SoSE), and consequently systems engineering, is currently being reformalized so as to provide greater functionality, integration, and extensibility into designed systems of systems (SoS). Graph theory and big data techniques are in position to play substantial roles in this reformalization, and hence, this paper fills a significant need by bringing together all known current applications of graph theory to SoSE in tutorial/summary form. In this paper, we explore the applications of graph theory and known graph algorithms for system design (and SoS design), optimization techniques, complexity measures, and novel graph algorithms for real-time deployment of SoS's and managing SoS design. The work is straightforward with several examples, and is meant to be a standalone document that can be used to quickly come up to speed regarding the applications of graph theory in SoSE.

Keywords:-

Eulerian Graph, Tree, Kur-atowski's theorem, Hamiltonian Graph , Chromatic number. Directed Graph , Planer Graph, Connectivity, Weighted Graph .

1 Introduction

Graph theory is a branch of mathematics that deals with the study of graphs, which are mathematical structures used to model relationships between objects. A graph consists of vertices (nodes) and edges (connections), making it a powerful tool for representing complex systems in various fields.

The significance of graph theory lies in its ability to provide solutions for problems related to connectivity, optimization, and network analysis. It has wide-ranging applications in computer science (e.g., algorithms, data structures, and artificial intelligence), biology (e.g., genetic networks

and neural connections), social sciences (e.g., social network analysis and behavioral studies), and engineering (e.g., communication and transportation networks).

As technology advances, the role of graph theory continues to expand, contributing to efficient problem-solving in fields such as big data analytics, cybersecurity, and quantum computing. This paper

explores the fundamental principles of graph theory, its real-world applications, and its impact on modern scientific and technological advancements.

System of systems engineering (SoSE) is currently undergoing a bit of a revolution so as to provide systems of systems (SoS) that are more robust, adaptable, extensible, etc. Key aspects of this revolution are modeling and simulation in big data environments, and some of these techniques rely on graph theoretical interpretations of the systems themselves as well as entities involved in designing, building, and deploying systems within the SoS framework. Although the application of graph theory to the SoS environment is not new (see [1]–[3]), theorems and algorithms beyond those used to this point can still be leveraged for enhancing the design, implementation, and policy environments when working with SoS. In this paper, we outline several challenges in SoSE, and propose graph theoretical tools and constructs for addressing some of these issues. We discuss graph theoretical algorithms, as well as graph representations to aid the policy and planning side of complex SoS design. The paper provides both an outline of existing literature, as well as proposals of new ways for thinking about SoSE using a graph-based approach. Thus, the intent is that this work will cover all current concepts of graph theory used for SoSE in addition to some novel contributions by the author.

2 History

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Following Euler's work, graph theory remained relatively unexplored until the 19th and early 20th centuries, when mathematicians such as Gustav Kirchhoff applied it to electrical circuit analysis and Arthur Cayley used it in chemical structures. The field gained further momentum with the work of Dénes Kőnig, who published one of the first comprehensive books on graph theory in 1936.

During the mid-to-late 20th century, graph theory became a crucial tool in computer science, operations research, and network analysis. The development of algorithms, such as Dijkstra's algorithm for shortest paths and the PageRank algorithm for web search, demonstrated its practical applications.

Today, graph theory continues to be a vital area of research, influencing artificial intelligence, machine learning, bioinformatics, and social network analysis. Its evolution from a theoretical problem to a widely applicable mathematical tool showcases its growing importance in solving real-world challenges.

The problem, which was about finding a path that crossed each of the seven bridges in the city of Königsberg (now Kaliningrad, Russia) exactly once, led Euler to develop the first theorem in graph theory. Euler's solution involved representing the landmasses as points (vertices) and the bridges as lines (edges) connecting them, effectively creating the first visual representation of a modern graph. While Euler's work laid the foundation, the term "graph theory" emerged later, in 1878, with the work of British mathematician Arthur Cayley. Graph theory was initially used to solve problems in areas like map coloring (the Four Color Theorem) and network analysis. Graph theory has expanded to a wide range of fields, including computer science, social networks, biology, and transportation planning. Graph theory remains an active area of research, with ongoing developments in areas like network analysis, algorithms, and applications in various disciplines.

3 Preliminaries of Graph Theory

Before analyzing the role of graph theory, it is essential to understand its fundamental concepts and definitions. These preliminaries form the basis for studying graphs and their applications. Graph- A graph is denoted as $G(V,E)$ graph consisting of two set vertices 'V' and edges 'E'. In Mathematics, a graph is a pictorial representation of any data in an organised manner. The graph shows the relationship between variable quantities. In a graph theory, the graph represents the set of objects, that are related in some sense to each other. The objects are basically mathematical concepts, expressed by vertices or nodes and the relation between the pair of nodes, are expressed by edge.

3.1 Basic Definitions:

- Graph :- A graph G consists of set of vertices V and a set of edges E that connect pairs of vertices. Formally, $G=(V,E)$. graph can be categorized based on their structure and properties.
- Directed Graph (Digraph):- In a directed graph, each edge has a direction, indicated by an arrow. An edge from vertex u to vertex v means you can traverse from u to v , but not necessarily from v to u .
- Weighted Graph:- A weighted graph assigns a numerical value (weight) to each edge, representing quantities like distance, cost, or capacity. This is particularly useful in problems involving optimization, such as finding the shortest path between two points.
- Simple Graph:- A simple graph has no loops (edges connecting a vertex to itself) and no multiple edges between the same pair of vertices.
- Multigraph:- A multigraph allows multiple edges between the same pair of vertices and may include loops.
- Undirected Graph:- In an undirected graph, edges have no orientation; they simply connect two vertices symmetrically. For example, if there's an edge between vertices u and v , you can traverse from u to v and vice versa.

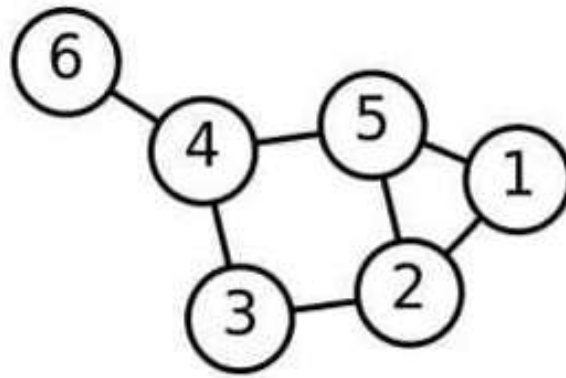


Figure 1: Undirected Graph

- **Bipartite Graph:-** A bipartite graph's vertex set can be divided into two disjoint subsets such that no two graph vertices within the same subset are adjacent. This type of graph is commonly used to model relationships between two different classes of objects, like jobs and workers.
- **Cycle Graph:-** A cycle graph consists of a single cycle, meaning a path of edges and vertices wherein a vertex is reachable from itself. It's a closed loop with no other connections.
- **Tree:-** A tree is an acyclic connected graph, meaning it has no cycles and there is exactly one path between any two vertices. Trees are fundamental in data structures and algorithms, representing hierarchical relationships.
- **Complete Graph:-** A complete graph is one in which every pair of distinct vertices is connected by a unique edge. In a complete undirected graph with n vertices, there are $\frac{n(n-1)}{2}$ edges.

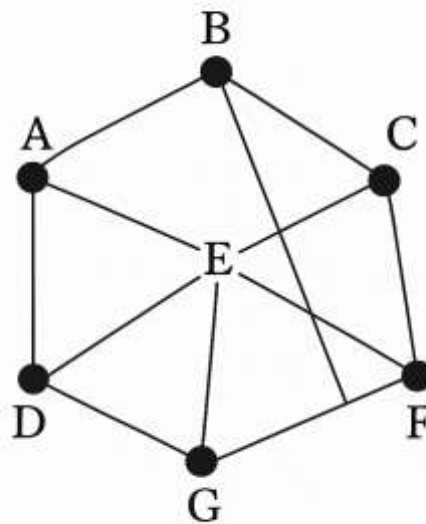


Figure 2: Complete Graph

- **Regular Graph:-** A graph G is said to be regular if every vertex has the same degree. If degree of each vertex of graph G is K , then it is called k -regular graph.

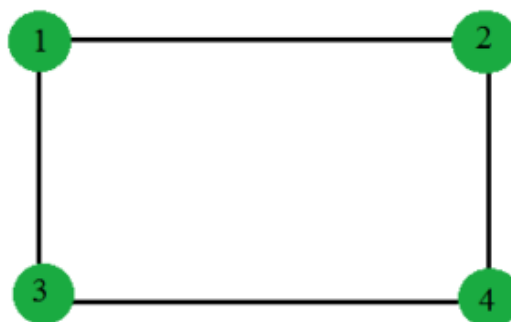


Figure 3: Regular Graph

3.2 Fundamental Concepts:

- **Degree:-** The degree of a vertex is the number of edges incident to it. In a directed graph, we distinguish between in-degree (number of incoming edges) and out-degree (number of outgoing edges).
- **Path:-** A path is a sequence of vertices where each adjacent pair is connected by an edge. A path is simple if it doesn't repeat any vertices.
- **Cycle:-** A cycle is a path that starts and ends at the same vertex without repeating any other vertices.
- **Connected Graph:-** A graph is connected if there is a path between every pair of vertices. In directed graphs, if there is a directed path from any vertex to every other vertex, the graph is strongly connected.
- **Subgraph:-** A subgraph is a subset of a graph's vertices and edges that forms a graph. A spanning subgraph includes all the vertices of the original graph.

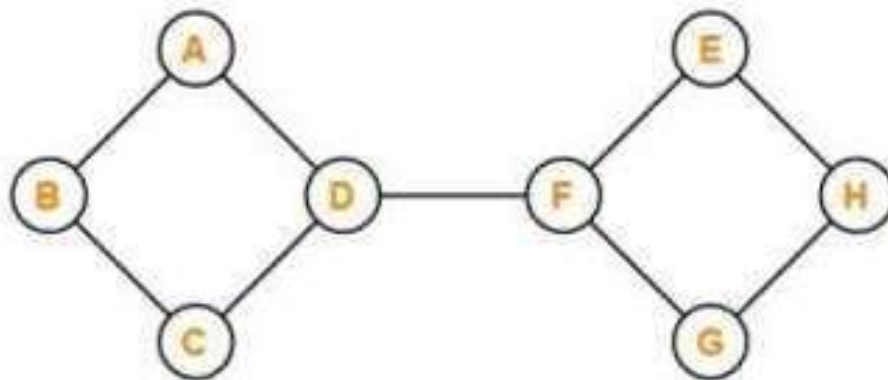


Figure 4: Connected Graph

- **Isomorphism:-** Two graphs are isomorphic if there's a one-to-one correspondence between their vertex sets and edge sets that preserves adjacency. Isomorphic graphs are structurally identical, differing only in the labeling of vertices and edges.

Understanding these preliminaries provides a solid foundation for exploring more advanced topics in graph theory, such as graph algorithms, network flows, and graph coloring.

3.3 Complement of graph:-

In graph theory, the complement of a graph G is a graph $\bar{G}$ that shares the same set of vertices as G but includes an edge between two vertices if and only if those vertices are not adjacent in G . In other words, $\bar{G}$ contains precisely the edges that are absent in G .

4 Key Properties of Graph Complements:

- **Planer graph:-** In graph theory, a planar graph is a graph that can be drawn on a plane without any of its edges crossing. This means it's possible to represent the graph in such a way that edges only intersect at their endpoints.

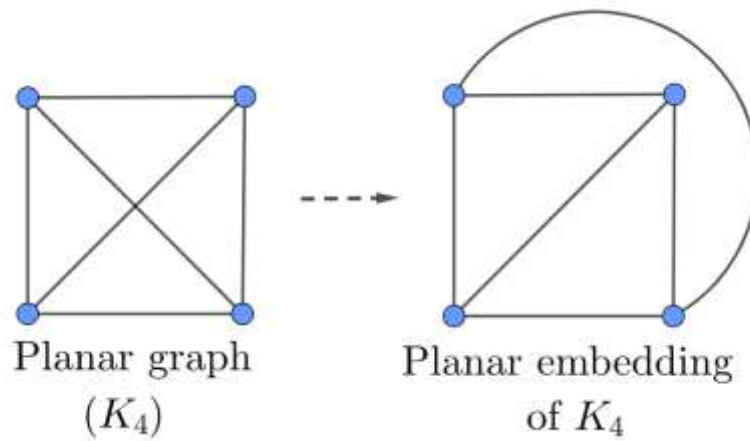


Figure 5: Caption

- **Binary tree:-** In computer science, a binary tree is a hierarchical data structure where each node has at most two children, referred to as the left child and the right child. This structure is widely used due to its efficient data storage and retrieval capabilities.

- **Rooted tree:-** Rooted tree is a tree in which one vertex is root.

- **Path length of tree:**

1. **Edge Inversion:-** An edge exists between two vertices in G if and only if it does not exist between those vertices in G .

2. **Vertex Consistency:-** G and G have identical vertex sets.

3. **Order:-** The number of vertices (order) remains unchanged between G and G .

Self-Complementary Graphs:-

A graph is termed self-complementary if it is isomorphic to its complement, meaning there exists a one-to-one correspondence between the vertices of the graph and its complement that preserves adjacency. Notable examples include.

- The four-vertex path graph.
- The five-vertex cycle graph.

5 Applications and Theorems:

The concept of graph complements is instrumental in various areas of graph theory. For instance, the Perfect Graph Theorem states that a graph is perfect if and only if its complement is also perfect. A perfect graph is one in which the chromatic number of every induced subgraph equals the size of the largest clique in that subgraph. This theorem has implications in optimization and algorithm design.

1. **Computer Science:-**

- **Networking:-** Internet and LAN structures use graph theory to model communication.

- Algorithms:-Graph search (DFS, BFS), shortest paths (Dijkstra's), and spanning trees (Kruskal's, Prim's).
 - Compilers:- Dependency resolution and syntax trees.
2. Social Networks:- Represent people as nodes and relationships as edges to analyze influence, centrality, or communities.
3. Biology:-
- Genomics:-DNA sequencing uses de Bruijn graphs.
 - Neuroscience:-Brain connectivity graphs.

Operations Research

- Transportation and Logistics:- Routing problems such as the Traveling Salesman Problem (TSP) and the Vehicle Routing Problem.
 - Project Planning:-Critical Path Method (CPM) and PERT charts.
4. Linguistics and Search Engines
- Natural Language Processing:-Word graphs and semantic nets.
 - PageRank Algorithm:-Google's ranking algorithm is based on eigenvectors of graph matrices.
 - Epidemiology:- Graph theory plays a crucial role in modelling the spread of diseases. Nodes represent individuals, and edges represent contacts or interactions between them. Epidemiologists use graph algorithms to simulate disease transmission, identify key influencers, and develop strategies for disease control.
 - Power Grids:- Graph theory is employed in modelling and analyzing electrical power grids. Nodes represent power stations, substations, and consumers, while edges represent transmission lines. Graph algorithms help in optimizing power flow identifying vulnerabilities, and designing resilient power grid systems.
 - Game Theory:- Graph theory is applied in graph theory to model strategic interactions between players in various games, including social dilemmas, voting systems, and economic competitions. Graph algorithms help analyze equilibrium strategies, coalition formations, and game dynamics.
 - Genetics:- Graph theory is used in genetics for analyzing genetic networks, genome sequencing, and evolutionary relationship between species. Graph algorithms help in identifying genetic patterns, predicting gene functions, and understanding genetic diseases.
 - Geography:- Graph theory is used in geographic information systems (GIS) for spatial analysis, route planning, and mapping. It helps in analyzing spatial relationships and optimizing geographic data.

6 Important Theorems in Graph Theory

1. Euler's Theorem (Eulerian Path and Circuit)

- A graph has an Eulerian circuit if every vertex has an even degree and the graph is connected.
- A graph has an Eulerian path if exactly two vertices have odd degrees.

2. Handshaking Lemma:-

- In any undirected graph, the sum of all Vertex degrees is twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

3. Kuratowski's Theorem

- A graph is non-planar if and only if it contains a subgraph that is a subdivision of K_5 or $K_{3,3}$.

4. König's Theorem:-

- In bipartite graphs, the size of the maximum matching equals the size of the minimum vertex cover.

5. Menger's Theorem:-

- Relates the maximum number of disjoint paths between two vertices to the minimum number of vertices whose removal disconnects them.

6. Four Color Theorem:-

- Every planar graph (or equivalently, every map) can be colored with at most four colors so that no adjacent regions share a color.

Theorem: Kuratowski's second graph is also nonplanar.

Proof:- Several properties common to the two graphs of Kuratowski. These are

1. Both are regular graphs.
2. Both are nonplanar.

Both are nonplanar.

3. Removal of one edge or a vertex makes each a planar graph.

4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the Smallest number of edges. Thus both are the simplest nonplanar graphs. In the literature, Kuratowski's first graph is usually denoted by K_5 and the Second graph by $K_{3,3}$ —letter K being for Kuratowski.

Graph theory serves as a fundamental cornerstone in mathematics and computer science, offering powerful tools to model and solve complex problems involving relationships and networks. Its versatility allows for broad applications across diverse fields, including computer networking, social sciences, biology, logistics, and artificial intelligence. From optimizing transportation systems to analyzing social networks and mapping molecular structures, graph theory not only enhances our understanding of connections but also enables efficient solutions to real-world challenges. As technology advances and data becomes more

interconnected, the importance and scope of graph theory will continue to expand, solidifying its role in both theoretical research and practical innovation.

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8 Conclusion

Graph theory plays a fundamental role in discrete mathematics, offering powerful tools and concepts for analyzing and solving a wide range of problems. Through its study, mathematics and researchers gain insights into the structure, connectivity, and properties of networks and discrete structures. The applications of graph theory extend across numerous fields, including computer science, biology, telecommunications, geography, and social sciences, making it a versatile and indispensable area of study. With its rich theoretical foundations and practical applications, graph theory continues to inspire new discoveries and innovations, shaping our understanding of complex systems and networks in the digital age and beyond. Graph theory's versatility and applicability make it an indispensable tool in both theoretical research and practical problem-solving. Its ability to abstract and analyze complex interconnections allows for advancements in technology, science, and social understanding. As we continue to face intricate challenges in various fields, the principles of graph theory will remain central to developing innovative solutions.

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