# BCK/BCH Algebras Associated with Pyramid Graphs

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***Abstract*- Open Pyramid Graph has been defined and identified by BCH/BCK algebras in [5]. In this paper a closed and a one door pyramid graphs have been identified by positive BCH and BCK algebras. Further, a closed pyramid graph has been extended and converted into a one door pyramid graph via positive BCH and BCK algebras.**

***Keywords*: Pyramid Graphs, BCK/BCH algebra.**

## §1. Introduction

**Definition (1.1) :** A structure (L , •, 0) where L is non empty set, '•' is a binary operation and 0 is a fixed element (also called zero element) is called a BCK algebra [1,2] if the elements of L satisfy the conditions:

[B1] $0 \bullet l = 0 \ \ for\ all\ l \in L;$
[B2] $\big((l \bullet m) \bullet (l \bullet n)\big) \bullet (n \bullet m) = 0;$
[B3] $\big(l \bullet (l \bullet m)\big) \bullet m = 0;$
[B4] $l \bullet l = 0;$
[B5] $l \bullet m = 0\ and\ \ m \bullet l = 0 \Rightarrow l = m.$

**Lemma (1.2) :** In a BCK algebra (L , • , 0) the following conditions are also satisfied :

[B6] $l \bullet 0 = l;$
[B7] $(l \bullet m) \bullet n = (l \bullet n) \bullet m;$
[B8] $l \bullet \big(l \bullet (l \bullet m)\big) = l \bullet m.$

**Definition (1.3) :** A system (L , • , 0) is called

a) a BCH algebra [3], if the elements of L satisfy only conditions [B4], [B5] and [B7] ;
b) a positive BCH algebra, if L is a BCH algebra with additional condition [B1].

**Definition (1.4) :** Let (L , • , 0) be a BCK/Positive BCH algebra. Then

a) two distinct elements $l$ and m of L are said to be mutually disjoint if $l \bullet$ m = $l$ and m $\bullet\ l = m$;
b) $l$ and m are said to be semi mutually disjoint if $l \bullet$ m = $l$ and m $\bullet\ l$ = 0 (or m $\bullet\ l$ = m and $l \bullet$ m = 0).

**Definition (1.5):** A pyramid is a simple graph with one vertex point, say $l_0$, n base points $l_1, l_2, l_3, \ldots, l_n$ and n edges connecting $l_0$ to $l_1, l_2, l_3, \ldots, l_n$. Such pyramid graph is of order n +1.

**Definition (1.6):** A pyramid graph is said to be:

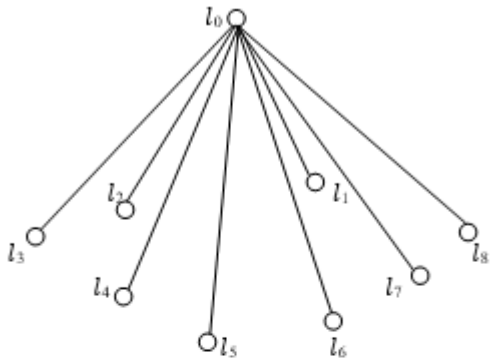
a) closed pyramid, if adjacent base points are connected by an edge.
b) one door pyramid, if every point is connected to its adjacent point except two adjacent points. If $l_k$ and $l_{k+1}$ are not connected we say that it is one door pyramid with door at $l_k$.

The above concept can also be explained as follows:

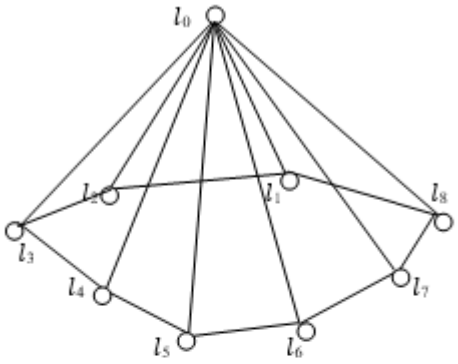
A simple graph with vertex point $l_0$ and edge set E is

a) a pyramid graph if d($l_0$) = n, d($l_i$) = 1 for $i$ = 1, 2, 3,…., n and E = $\{l_0l_1, l_0l_2, \ldots, l_0l_n\}$. This graph is denoted by $P_y(l_0; l_1, l_2, l_3, \ldots, l_n)$.
b) a closed pyramid graph, if d($l_0$) = n, d($l_i$) = 3 for $i$ = 1, 2, 3,….,n and E = $\{l_0l_1, \ldots, l_0l_n, l_1l_2, l_2l_3 \ldots, l_{n-1}l_n, l_nl_1\}$. We denote such graph as $clP_y(l_0; l_1, l_2, l_3, \ldots, l_n)$.
c) a one door pyramid graph if there exists k with 1≤ k ≤ n such that d($l_0$) = n, d($l_i$) = 3

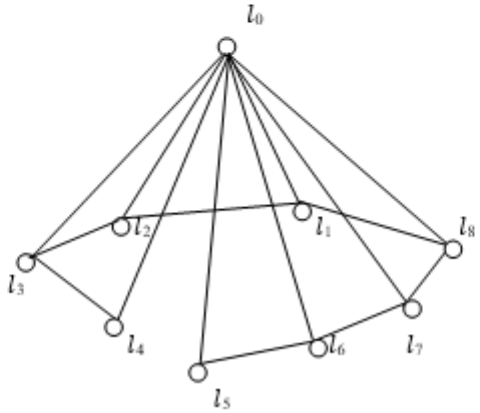
for $i$ = 1, 2,…., k-1, k+2,…., n, d($l_k$) = 2, d($l_{k+1}$) = 2 and E = {$l_0l_1$,….,$l_0l_n$,$l_1l_2$, $l_2l_3$, ….,$l_{k1}l_k$, $l_{k+1}l_{k+2}$,…., $l_nl_1$}. This is denoted by od$P_y$($l_0$; $l_1$, $l_2$, $l_3$, …., $l_n$).
Such graphs are represented below:




Pyramid Graph




Closed Pyramid Graph




One door Pyramid Graph

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## §2. PYRAMID GRAPHS AND BCK/BCH ALGEBRAS

Here we have developed some results connecting pyramid graphs with BCK/ BCH algebras.

**Theorem (2.1) :-** Let cl$P_y$($l_0$; $l_1$, $l_2$, $l_3$, …., $l_n$) be a closed pyramid graph of order n+1. Then there exist binary operations '•' and '△' on the set L = {$l_0$, $l_1$, $l_2$, $l_3$, …., $l_n$} such that (L,•, $l_0$) and (L, △, $l_0$) are respectively positive BCH and BCK algebras. Further, the graph of disjoint elements of these algebraic structures coincide with the given closed pyramid graph.

**Proof:-** Let L= {$l_0$, $l_1$, $l_2$, $l_3$, …., $l_n$} and let a binary operation '•' be defined on L as follows:

i) '•' satisfies the conditions :

$$l_0 \bullet l_i = l_0; \quad l_i \bullet l_0 = l_i, \quad i = 1, 2, \ldots, n \text{ and } l_i \bullet l_i = l_0 \text{ for } i = 0, 1, 2, \ldots, n; \tag{2.1}$$

ii) $l_i \bullet l_{i+1} = l_i$; $l_{i+1} \bullet l_i = l_{i+1}$ for $i$ = 1, 2, …., n–1 and $l_n \bullet l_1 = l_n$; $l_1 \bullet l_n = l_1$; (2.2)

iii) for $i$ = 1,2,….,n–1; j $\neq i, i+1$ and for $i$ = n, j = 1 let

either $l_i \bullet l_j = l_i$, $l_j \bullet l_i = l_0$

or $l_i \bullet l_j = l_0$, $l_j \bullet l_i = l_j$. (2.3)

Now we prove that (L,•, $l_0$) is a positive BCH –algebra. We see that [B1], [B4] and [B5] are satisfied from the given conditions. In order to prove [B7], consider triplets $l_i$, $l_{i+1}$ and $l_j$ (j $\neq i, i+1$) where $l_i \bullet l_j = l_i$, $l_j \bullet l_i = l_0$; $l_{i+1} \bullet l_j = l_0$, and $l_j \bullet l_{i+1} = l_j$.

Then

$(l_i \bullet l_{i+1}) \bullet l_j = l_i \bullet l_j = l_i$; $(l_i \bullet l_j) \bullet l_{i+1} = l_i \bullet l_{i+1} = l_i$;

$(l_{i+1} \bullet l_i) \bullet l_j = l_{i+1} \bullet l_j = l_0$; $(l_{i+1} \bullet l_j) \bullet l_i = l_0 \bullet l_i = l_0$;

$(l_j \bullet l_i) \bullet l_{i+1} = l_0 \bullet l_{i+1} = l_0$; $(l_j \bullet l_{i+1}) \bullet l_i = l_j \bullet l_i = l_0$.

In this way we can prove that [B7] is satisfied for all triplets of L. This proves that (L,•, $l_0$) is a positive BCH-algebra. Also pairs of mutually disjoint elements in this BCH-algebra are {$l_0$,$l_1$},….,{$l_0$,$l_n$}, {$l_1$,$l_2$}, {$l_2$, $l_3$},….,{$l_{n-1}$, $l_n$} and {$l_n$, $l_1$}.

Suppose a binary operation '△' satisfies conditions (2.1), (2.2) of binary operation '•' and

**(iv)** $l_i \triangle l_j = l_i$, $l_j \triangle l_i = l_0$ for rest of the suffixes $i, j$ with $i < j$. (2.4)

Conditions [B1], [B4] and [B5] for a BCK algebra are satisfied from given conditions. So we have only to prove conditions [B2] and [B3]. We see that

$$(l_i \triangle (l_i \triangle l_{i+1})) \triangle l_{i+1} = (l_i \triangle l_i) \triangle l_{i+1} = l_0 \triangle l_{i+1} = l_0.$$

For $i < j$,

$$(l_i \triangle (l_i \triangle l_j)) \triangle l_j = (l_i \triangle l_i) \triangle l_j = l_0 \triangle l_j = l_0;$$
$$(l_j \triangle (l_j \triangle l_i)) \triangle l_i = (l_j \triangle l_0) \triangle l_i = l_j \triangle l_i = l_0.$$

Thus [B3] is satisfied for all elements of L.

We consider triplets $l_i$, $l_{i+1}$ and $l_j$ where $i + 1 < j$.

Then

$$((l_i \triangle l_{i+1}) \triangle (l_i \triangle l_j)) \triangle (l_j \triangle l_{i+1}) = (l_i \triangle l_i) \triangle l_0 = l_0;$$

and $((l_{i+1} \triangle l_i) \triangle (l_{i+1} \triangle l_j)) \triangle (l_j \triangle l_i) = (l_{i+1} \triangle l_{i+1}) \triangle l_0 = l_0 \triangle l_0 = l_0.$

Taking other triplet we see that [B2] is satisfied. Hence (L, △, $l_0$) is a BCK algebra in which the elements of the pairs $\{l_0, l_1\}, \{l_0, l_2\}, \ldots, \{l_0, l_n\}, \{l_1, l_2\}, \ldots, \{l_{n-1}, l_n\}$ and $\{l_n, l_i\}$ are mutually disjoint.

This means that in the both cases the graphs of mutually disjoint elements are coincident with the given closed pyramid graph.

**Theorem (2.2):-** Let $OdP_y(l_0; l_1, l_2, l_3, \ldots, l_k, l_{k+1}, \ldots, l_n)$ be a one door pyramid graph with door at $(l_k, l_{k+1})$. Then there exist binary operations '•' and '△' on the set $L = \{l_0, l_1, l_2, l_3, \ldots, l_n\}$ such that (L, •, $l_0$) and (L, △, $l_0$) are respectively positive BCH and BCK algebras. Also, the graphs of mutually disjoint elements of these algebras coincide with the given one door pyramid graph.

**Proof:-** Let $L = \{l_0, l_1, l_2, l_3, \ldots, l_n\}$ and let a binary operation '•' be defined on L with following conditions:

a) '•' satisfies the condition (2.1).

b) $l_i \bullet l_{i+1} = l_i$; $l_{i+1} \bullet l_i = l_{i+1}$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n-1$; $l_n \bullet l_1 = l_n$; $l_1 \bullet l_n = l_1$; (2.5)

*c*) $l_k \bullet l_{k+1} = l_k$; $l_{k+1} \bullet l_k = l_0$ (2.6)

d) Condition (2.3) for rest of the suffixes.

As previous results we see that [B1], [B4] and [B5] are satisfied for a BCH algebra. To prove condition [B7] we consider triplets $l_k$, $l_{k+1}$ and $l_j$ $(j \neq k, k+1)$ and assume that $l_k \bullet l_j = l_k$, $l_j \bullet l_k = l_0$, $l_{k+1} \bullet l_j = l_{k+1}$, $l_j \bullet l_{k+1} = l_0$.

Then

$$(l_k \bullet l_{k+1}) \bullet l_j = l_k \bullet l_j = l_k; \qquad (l_k \bullet l_j) \bullet l_{k+1} = l_k \bullet l_{k+1} = l_k;$$
$$(l_{k+1} \bullet l_k) \bullet l_j = l_0 \bullet l_j = l_0; \qquad (l_{k+1} \bullet l_j) \bullet l_k = l_{k+1} \bullet l_k = l_0;$$
$$(l_j \bullet l_k) \bullet l_{k+1} = l_0 \bullet l_{k+1} = l_0; \qquad (l_j \bullet l_{k+1}) \bullet l_k = l_0 \bullet l_k = l_0.$$

For other triplets, [B7] can be proved as above. So (L, •, $l_0$) is a positive BCH algebra. Also pairs of disjoint elements in this BCH algebra are $\{l_0, l_1\}, \{l_0, l_2\}, \ldots, \{l_0, l_n\}, \{l_1, l_2\}, \ldots, \{l_{k-1}, l_k\}, \{l_{k+1}, l_{k+2}\}, \ldots, \{l_{n-1}, l_n\}$ and $\{l_n, l_i\}$.

Suppose binary operation '△' satisfies conditions (a), (b) and (c) stated above for binary operation '•'. Further, '△' satisfies conditions.

e) $l_i \triangle l_j = l_i$ and $l_j \triangle l_i = l_0$ for rest of the suffixes $i, j$ with $i < j$. (2.7)

Conditions [B1], [B4] and [B5] for a BCK algebra are satisfied from given conditions. So, in order to prove that L is a BCK algebra with respect to binary operation '△', it remains to prove conditions [B2] and [B3]. We see that

$$(l_{k+1} \triangle (l_{k+1} \triangle l_k)) \triangle l_k = (l_{k+1} \triangle l_0) \triangle l_k = l_{k+1} \triangle l_k = l_0.$$

Now for $k+1 < j$,

$$(l_{k+1} \triangle (l_{k+1} \triangle l_j)) \triangle l_j = (l_{k+1} \triangle l_{k+1}) \triangle l_j = l_0 \triangle l_j$$

and $((l_{k+1} \triangle l_k) \triangle (l_{k+1} \triangle l_j)) \triangle (l_j \triangle l_k) = (l_0 \triangle l_{k+1}) \triangle l_0 = l_0.$

In this way we can prove that [B2] and [B3] are satisfied for all elements of L including the case in which $j < k$.

This proves that (L, △, $l_0$) is a BCK algebra. Also the set of mutually disjoint elements in this BCK algebra is same as that stated for BCH algebra and the graph associated with mutually disjoint elements is the given one door pyramid graph.

## §3. Extension and Conversion of Pyramid Graph

**Theorem (3.1) :-** Every closed pyramid graph can be extended to a one door pyramid graph via:

a) A positive BCH algebra b) a BCK algebra

**Proof :-** Let $clP_y(l_0; l_1, l_2, l_3, \ldots, l_n)$ be a closed pyramid graph and let $L = \{l_0, l_1, l_2, l_3, \ldots, l_n\}$.

(a) By theorem (2.1) there exists a positive BCH algebra (L, •, $l_0$) such that the graph of mutually disjoint elements of this BCH algebra coincides with the given closed pyramid graph. Let $l_t$ be a point outside L and this point be located between $l_n$ and $l_1$. We assume $L_1 = L \cup \{l_t\}$. Let a binary operation '$\bullet_,$' be defined on $L_1$ as follows:

$l_i \bullet_, l_j = l_i \bullet l_j$ for all $i$, j except $i$ = n, j = 1,

$l_n \bullet_, l_1 = l_0$ (3.1)

$l_n \bullet_, l_t = l_n$, $l_t \bullet_, l_n = l_t$ (3.2)

$l_t \bullet_, l_1 = l_t$, $l_1 \bullet_, l_t = l_0$ (3.3)

$l_0 \bullet_, l_t = l_0$, $l_t \bullet_, l_0 = l_t$ (3.4)

$l_i \bullet_, l_t = l_i$, $l_t \bullet_, l_i = l_0$ ; $i$ = 1, 2, …….., n-1 (3.5)

Now we prove that ($L_1$, $\bullet_,$, $l_0$) is a positive BCH algebra. It is easy to verify conditions [B1], [B4] and [B5]. To prove condition [B7] we consider triplet $\{l_n, l_t, l_1\}$. We see that

$(l_n \bullet_, l_t) \bullet_, l_1 = l_n \bullet_, l_1 = l_0$; $(l_n \bullet_, l_1) \bullet_, l_t = l_0 \bullet_, l_t = l_0$.

For triplet $\{l_n, l_t, l_i\}$, $i$ = 1, 2, …., n-1, we have,

$(l_n \bullet_, l_t) \bullet_, l_i = l_n \bullet_, l_i = l_n \bullet l_i$; $(l_n \bullet_, l_i) \bullet_, l_t = (l_n \bullet l_i) \bullet_, l_t = l_n \bullet l_i$.

Thus the condition [B7] for BCH algebra is satisfied for this binary operation. This proves that ($L_1$, $\bullet_,$, $l_0$) is a BCH algebra. The graph associated with mutually disjoint elements is a one door pyramid graph with door at $l_t$.

(b) Again there exists a binary operation '△' on L such that (L, △, $l_0$) is a BCK-algebra [theorem (2.1)] and the graph associated with mutually disjoint elements of L is the given closed pyramid graph. Let $l_t$ be a point between $l_n$ and $l_1$. Let $\triangle_,$ be a binary operation defined on $L_1 = L \cup \{l_t\}$ as given for '$\bullet_,$' in conditions (3.1) to (3.5) with replacement of '$\bullet_,$' by '$\triangle_,$' in (3.1) for others remaining suffixes.

In order to prove that ($L_1$, $\triangle_,$, $l_0$) is a BCK algebra we see that (B1), (B4) and (B5) are satisfied. We need to check conditions (B2) and (B3) for certain specified points. We consider triplets $\{l_n, l_t, l_i$ with $i < n\}$.

Then $((l_n \triangle_, l_t) \triangle_, (l_n \triangle_, l_i)) \triangle_, (l_i \triangle_, l_t) = (l_n \triangle_, l_0) \triangle_, l_i = (l_n \triangle_, l_i) = l_0$;

$((l_t \triangle_, l_n) \triangle_, (l_t \triangle_, l_i)) \triangle_, (l_i \triangle_, l_n) = (l_t \triangle_, l_0) \triangle_, l_i = (l_t \triangle_, l_i) = l_0$;

$((l_i \triangle_, l_t) \triangle_, (l_i \triangle_, l_n)) \triangle_, (l_n \triangle_, l_t) = (l_i \triangle_, l_i) \triangle_, l_n = l_0$.

Further, $(l_t \triangle_, (l_t \triangle_, l_n)) \triangle_, l_n = (l_t \triangle_, l_t) \triangle_, l_n = l_0 \triangle_, l_n = l_0$;

$(l_t \triangle_, (l_t \triangle_, l_i)) \triangle_, l_i = (l_t \triangle_, l_0) \triangle_, l_i = l_t \triangle_, l_i = l_0$.

In this way (B2) and (B3) can be proved for other elements of $L_1$. So ($L_1$, $\triangle_,$, $l_0$) is a BCK algebra such that graph associated with mutually disjoint elements of $L_1$ is a one door pyramid graph with door at $l_t$.

**Theorem (3.2):-** Every closed pyramid graph can be converted into one door pyramid by redefining the binary operations but retaining the structure.

**Proof:-** Suppose that $clP_y$ ($l_0$; $l_1$, $l_2$, $l_3$, …., $l_n$) is a closed pyramid graph.We assume that we want to convert this graph into a one door pyramid graph with door at $l_k$.

As discussed in theorem (2.1) there exists a positive BCH algebra (L,•, $l_0$) such that the graph of mutually disjoint elements of the BCH algebra coincides with the given closed pyramid graph. We consider the point $l_k$. We redefine a binary operation '$\bullet_,$' on L as follows:

$l_i \bullet_, l_j = l_i \bullet l_j$ for $i$, j = 1, 2,……, k, k+1,……,n (3.6)

$l_k \bullet_, l_{k+1} = l_k$, $l_{k+1} \bullet_, l_k = l_0$ (3.7)

We prove that (L, $\bullet_,$, $l_0$) is a positive BCH-algebra. We see that (B1), (B4) and (B5) are satisfied for the above defined binary operation. We need to prove (B7). We consider the triplets $\{l_k, l_{k+1}, l_j, j \neq k, k+1\}$ such that

$l_k \bullet l_j = l_k$, $l_j \bullet l_k = l_0$, $l_{k+1} \bullet l_j = l_0$, $l_j \bullet l_{k+1} = l_j$ (3.8)

Then, $(l_k \bullet_, l_{k+1}) \bullet_, l_j = l_k \bullet_, l_j = l_k \bullet l_j = l_k$; $(l_k \bullet_, l_j) \bullet_, l_{k+1} = l_k \bullet_, l_{k+1} = l_k$;

$(l_{k+1} \bullet_, l_j) \bullet_, l_k = l_0 \bullet_, l_k = l_0$; $(l_{k+1} \bullet_, l_k) \bullet_, l_j = l_0 \bullet_, l_j = l_0$;

$(l_j \bullet_, l_k) \bullet_, l_{k+1} = l_0 \bullet_, l_{k+1} = l_0$; $(l_j \bullet_, l_{k+1}) \bullet_, l_k = l_j \bullet_, l_k = l_0$.

This means that (B7) is satisfied. Hence (L, $\bullet_,$,$l_0$) is a positive BCH algebra such that the graph of mutually disjoint elements is a one door graph with door at $l_k$.

Further we see that (L, △, $l_0$) is a BCK algebra such that the graph of mutually disjoint elements coincide with the given closed pyramid graph. We want to have a door at the point $l_k$ (k < n). For this we redefine the binary operation △ as $\triangle_,$ as given in (3.6) and (3.7) and $\boldsymbol{l_i \triangle l_j = l_i, l_j \triangle l_i = l_0}$ **for** $\boldsymbol{i < j}$ . In order to prove that (L, $\triangle_,$,$l_0$) is a BCK algebra, we need to verify conditions (B2) and (B3) of BCK algebra. For this we consider triplet $\{l_k, l_{k+1}, l_j, j \neq k, k+1\}$ with j > k, k+1.

Then

$((l_j \triangle_, l_k) \triangle_, (l_j \triangle_, l_{k+1})) \triangle_, (l_{k+1} \triangle_, l_k) = (l_0 \triangle_, l_0) \triangle_, l_0 = l_0$

$((l_k \triangle_, l_{k+1}) \triangle_, (l_k \triangle_, l_j)) \triangle_, (l_j \triangle_, l_{k+1}) = (l_k \triangle_, l_k) \triangle_, l_0 = l_0$;

$((l_{k+1} \triangle_, l_k) \triangle_, (l_{k+1} \triangle_, l_j)) \triangle_, (l_j \triangle_, l_k) = (l_0 \triangle_, l_{k+1}) \triangle_, l_0 = l_0.$

Also $(l_k \triangle_, (l_k \triangle_, l_j)) \triangle_, l_j = (l_k \triangle_, l_k) \triangle_, l_j = l_0.$

If j < k, k+1, then

$((l_j \triangle_, l_k) \triangle_, (l_j \triangle_, l_{k+1})) \triangle_, (l_{k+1} \triangle_, l_k) = (l_j \triangle_, l_j) \triangle_, l_0 = l_0 \triangle_, l_0 = l_0;$

$((l_k \triangle_, l_{k+1}) \triangle_, (l_k \triangle_, l_j)) \triangle_, (l_j \triangle_, l_{k+1}) = (l_k \triangle_, l_0) \triangle_, l_j = l_k \triangle_, l_j = l_0;$

$((l_{k+1} \triangle_, l_k) \triangle_, (l_{k+1} \triangle_, l_j)) \triangle_, (l_j \triangle_, l_k) = (l_0 \triangle_, l_0) \triangle_, l_j = l_0 \triangle_, l_j = l_0.$

Also $(l_k \triangle_, (l_k \triangle_, l_j)) \triangle_, l_j = (l_k \triangle_, l_0) \triangle_, l_j = l_k \triangle_, l_j = l_0.$

This means that (B2) and (B3) are satisfied for the operation $\triangle_,$ in all cases. Hence $(L, \triangle_, l_0)$ is a BCK algebra such that the graph of mutually disjoint elements is a one door pyramid graph with door at $l_k$.

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