SOME STRUCTURAL PROPERTIES OF U- ALGEBRA

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***Abstract*- The concept of an U- algebra has been developed by Rashmi Rani and R.L Prasad in 2021[2]. Here we study some structural properties of U- algebras, methods to form U- algebras and extensions of U- algebras to U- algebras.**

Keyword: **U- algebra, Poset**

§1. Preliminaries

Definition (1.1): An U-algebra is a system (E, *, u) consisting of a non-empty set E, a binary operation * and a fixed element (also known as initial element) u subject to the following conditions:

(U 1) u * t = t;
(U 2) t * t = t;
(U 3) (l * s) * t = (l * t) * s;
(U 4) l * (s * t) = (l * s) * (l * t);

for all l, s, t ∈ E.

Lemma(1.2): In an U-algebra (E , *, u) the following conditions are satisfied[2].

(U 5) l * u = l;
(U 6) s * t = t * s;
(U 7) s * t = u ⇒ s = u and t = u;
(U 8) (s * (s * t)) * t = s * t;
(U 9) (s * t) * s = s * (s * t) = (t * s) * s = s * t;
(U 10) l * (s * t) = (l * s) * t.

For all s, t, l ∈ E.

Lemma(1.3): In an U-algebra (E, * ,u) a relation '≤' is defined as s ≤ t iff s * t = t * s = t ; s, t ∈ U. The relation '≤' defined above is a partial order relation.

Definition(1.4): Two elements s and t of E are said to be

(a) Comparable, if s * t = t * s = t (or s).

(b) Non- comparable, if s * t = t * s = l ≠ s or t.

(c) If there exists s ∈ E such that t ≤ s for all t ∈ E then s is called the maximal element of E. The minimal element in E is u.

Examples(1.5): We see that the singleton set E = {u} and doublet set E = { u , v} are U- algebras with initial element u and binary operations $*_1$ and $*_2$ defined as

u $*_1$ u = u and

$*_2$	u	v
u	u	v
v	v	v

If E = {u , s , t} then E become U- algebras under two binary operations $*_3$, $*_4$ given by tables

$*_3$	u	s	t
u	u	s	t
s	s	s	s
t	t	s	t

$*_3$	u	s	t
u	u	s	t
s	s	s	t
t	t	t	t

§2. Some properties

Theorem(2.1): An integral domain X with |X| ≥ 3 can not be an U- algebra under a binary operation defined by a quadratic expression.

Proof: Let X be an integral domain with $|X| \geq 3$, zero elements 0 and unit element 1. If possible, let (X , * , 0) be a quadratic system, which is an U- algebra.
Assume that x * y is expressible as

$$x * y = A x^2 + B x y + C y^2 + D x + E y + F, \quad (2.1)$$

where A , B , C , D , E , F ∈ X . Then condition (U 2) implies

$$x = x * x = A x^2 + B x^2 + C x^2 + D x + E x + F$$

$$\Rightarrow A + B + C = 0 , D + E = 1 , F = 0. \quad (2.2)$$

Again condition (U 5) implies

$$x = x * 0 = A x^2 + 0 + 0 + D x$$

$$\Rightarrow A = 0 , D = 1. \quad (2.3)$$

Condition (U 1) implies

$$x = 0 * x = 0 + C x^2 + E x$$

$$\Rightarrow C = 0 , E = 1. \quad (2.4)$$

Combing conditions (2.2) , (2.3) and (2.4), we get
A = B = C =0 , D + E = 1 , D = 1 , E = 1 , F = 0.
This is not possible. So an integral domain can not be an U- algebra under a quadratic binary expression.
Following results can be established easily

Theorem (2.2): Let (E , *, u) and (F, ● ,v) be U- algebras
and let H=E×F. Define a binary operation Θ is H as follows:
For (s , l) , (t , m) ∈ H , (s , l) Θ (t , m) = (s * t , l ● m). (2.5)
Then (H , Θ , (u , v)) is an U- algebra.

Theorem (2.3): If Cartesian product of two algebraic systems is an U– algebra then each system is an U- algebra.

Example (2.4): We recall U- algebra $(E, *_4 , u)$ from above
and let H = E × E = { a , b , c , d , e , f , g , h , i} where a = (u , u), b = (u , s), c=(u , t) , d=(s , u) , e=(s , s) , f=(s , t), g=(t , u) , h=(t , s) , i=(t , t).
Further we extend the binary operation '$*_4$' to binary operation Θ on H coordinate wise. The binary table for Θ is given by

Θ	a	b	c	d	e	f	g	h	i
a	a	b	c	d	e	f	g	h	i
b	b	b	c	e	e	f	h	h	i
c	c	c	c	f	f	f	i	i	i
d	d	e	f	d	e	f	g	h	i
e	e	e	f	e	e	f	h	h	i
f	f	f	f	f	f	f	i	i	i
g	g	h	i	g	h	i	g	h	i
h	h	h	i	h	h	i	h	h	i
i	i	i	i	i	i	i	i	i	i

Table (2.1)

Then by theorem (2.2)(H , Θ , (a , a)) is an U- algebra.

§3. Methods to form U- algebras

Theorem(3.1): Every finite set E with |E| > 4 can be made into an U- algebra under some suitable binary operations.

Proof: Let E = { $\lambda_0 , \lambda_1 , \lambda_2 , \ldots\ldots\ldots , \lambda_n$ } be a finite set. We take λ_0 as an initial element. Then E becomes an U-algebra under following binary operation.
Let $\lambda_k \neq \lambda_0$ be a fixed element of E. A binary operation Θ is defined on E as follows:

$\lambda_0 \Theta \lambda_i = \lambda_i \Theta \lambda_0 = \lambda_i$; i=1,…….,n; (3.1)

$\lambda_i \Theta \lambda_i = \lambda_i$; i=0,1,…….,n; (3.2)

$\lambda_k \Theta \lambda_i = \lambda_i \Theta \lambda_k = \lambda_k$; i=1,2,……,k-1,k+1,……..,n; (3.3)

$\lambda_i \Theta \lambda_j = \lambda_j \Theta \lambda_i = \lambda_k$; $i \neq j \neq k$. (3.4)

We prove that (E , Θ , λ_0) is an U- algebra.
From the given definitions (U 1) and (U 2) are satisfied. We need to check conditions (U 3) and (U 4). We consider following cases:
i) We choose element $\lambda_0, \lambda_i, \lambda_k \neq \lambda_i$
Then $(\lambda_k \Theta \lambda_0) \Theta \lambda_i = \lambda_k \Theta \lambda_i = \lambda_k$;
$(\lambda_k \Theta \lambda_i) \Theta \lambda_0 = \lambda_k \Theta \lambda_0 = \lambda_k$;

$\lambda_k \Theta (\lambda_0 \Theta \lambda_i) = \lambda_k \Theta \lambda_i = \lambda_k$;
$(\lambda_k \Theta \lambda_0) \Theta (\lambda_k \Theta \lambda_i) = \lambda_k \Theta \lambda_k = \lambda_k$.

ii) We take λ_i, λ_j, λ_k ; i ≠ j ≠ k, Then
$(\lambda_i \Theta \lambda_j) \Theta \lambda_k = \lambda_k \Theta \lambda_k = \lambda_k$;
$(\lambda_i \Theta \lambda_k) \Theta \lambda_j = \lambda_k \Theta \lambda_j = \lambda_k$;
$\lambda_i \Theta (\lambda_j \Theta \lambda_k) = \lambda_i \Theta \lambda_k = \lambda_k$;
$(\lambda_i \Theta \lambda_j) \Theta (\lambda_i \Theta \lambda_k) = \lambda_k \Theta \lambda_k = \lambda_k$.

Thus in both cases (U 3) and (U4) are satisfied. So (E , Θ , λ_0) is an U- algebra.

Corollary(3.2): We assume all the conditions of theorem(3.1) with following change:
Let $\lambda_s \neq \lambda_k$ be a fixed element of E, such that

$$\lambda_{n-1} \Theta \lambda_n = \lambda_n \Theta \lambda_{n-1} = \lambda_s \quad (3.5)$$

Then (E , Θ , λ_0) is an U- algebra.

Proof : We only need to check conditions (U 3) and (U 4) for elements λ_{n-1} , λ_n and $\lambda_i \neq \lambda_{n-1}, \lambda_n$. We have

$(\lambda_i \Theta \lambda_{n-1}) \Theta \lambda_n = \lambda_k \Theta \lambda_n = \lambda_k$;
$(\lambda_i \Theta \lambda_n) \Theta \lambda_{n-1} = \lambda_k \Theta \lambda_{n-1} = \lambda_k$;
$\lambda_i \Theta (\lambda_{n-1} \Theta \lambda_n) = \lambda_i \Theta \lambda_s = \lambda_k$;
$(\lambda_i \Theta \lambda_{n-1}) \Theta (\lambda_i \Theta \lambda_n) = \lambda_k \Theta \lambda_k = \lambda_k$;
$(\lambda_{n-1} \Theta \lambda_n) \Theta \lambda_i = \lambda_s \Theta \lambda_i = \lambda_k$;
$(\lambda_{n-1} \Theta \lambda_i) \Theta \lambda_n = \lambda_k \Theta \lambda_n = \lambda_k$;
$\lambda_{n-1} \Theta (\lambda_n \Theta \lambda_i) = \lambda_{n-1} \Theta \lambda_k = \lambda_k$;
$(\lambda_{n-1} \Theta \lambda_n) \Theta (\lambda_{n-1} \Theta \lambda_i) = \lambda_s \Theta \lambda_k = \lambda_k$.

This means that (U 3) and (U 4) are satisfied in all cases.
So, (E , Θ , λ_0) is an U- algebra.

Theorem(3.3) : Every ordered set E= {λ_0, λ_1…….. λ_n} an u- algebra under the binary operation $\lambda_i \Theta \lambda_j = \lambda_j \Theta \lambda_i = \lambda_j$ if $\lambda_i \leq \lambda_j$(i < j)

(3.6)

Proof: Clearly $\lambda_0 \Theta \lambda_i = \lambda_i \Theta \lambda_0 = \lambda_i$ for all i ≠ 0. Also $\lambda_i \Theta \lambda_i = \lambda_i$ for 0 ≤ i ≤ n,
For i < j < k we have

$(\lambda_i \Theta \lambda_j) \Theta \lambda_k = \lambda_j \Theta \lambda_k = \lambda_k$;
$(\lambda_i \Theta \lambda_k) \Theta \lambda_j = \lambda_k \Theta \lambda_j = \lambda_k$;
$\lambda_i \Theta (\lambda_j \Theta \lambda_k) = \lambda_i \Theta \lambda_k = \lambda_k$;
$(\lambda_i \Theta \lambda_j) \Theta (\lambda_i \Theta \lambda_k) = \lambda_j \Theta \lambda_k = \lambda_k$.

This proves that (X , Θ , λ_0) is an U- algebra .

Now we give a method to form an U- algebra on a set which consist of comparable and non- comparable elements.

Theorem(3.4): Let E be a finite set having some comparable and some non – comparable elements such that there exists elements u and l for which u ≤ s ≤ l for all s ∈ E. In view of lemma (1.3) a binary operation Θ is defined on E such that

u Θ s = s Θ u = s
s Θ s = s
s Θ t = t if s ≤ t
s Θ t = l if s and t are non – comparable

Then (E , Θ , u) is an U- algebra.

Proof: Conditions (U 1) and (U 2) are satisfied from the above assumptions. To establish (U 3) and (U 4) we choose following types of triplets :

i) Let triplet s, t, v be such that s ≤ t and v ≤ s. Then v≤ t. So we have
(s Θ t) Θ v = t Θ v = t, (s Θ v) Θ t = s Θ t = t;
t Θ (s Θ v) = t Θ s = t, (t Θ s) Θ (t Θ v) = t Θ t = t.

ii) Let s, t, v be triplets such that s and t are comparable and v is non- comparable with both s and t. Suppose that s * t = t. Then
(s Θ t) Θ v = t Θ v = l, (s Θ v) Θ t = l Θ t = l;
s Θ (t Θ v) = s Θ l = l, (s Θ t) Θ (s Θ v) = t Θ l = l.

iii)Let triplets s, t, v are pair wise non – comparable. Then
(s Θ t) Θ v = l Θ v = l, (s Θ v) Θ t = l Θ t = l;

s Θ (t Θ v) = s Θ l = l, (s Θ t) Θ (s Θ v) = l Θ l = l.
so, in all cases (U 3) and (U 4) are satisfied, Hence the result.

§ 4. Extensions of U- algebra

The following results can be proved easily.

Theorem(4.1): Let (E , * , θ) be an U- algebra and let δ ∉ E. Let Y= E U {δ} and define a binary operation ● on Y as follows:

$$\lambda \bullet \mu = \lambda * \mu \quad \text{if } \lambda, \mu \in E \tag{4.1}$$
$$\theta \bullet \delta = \delta \bullet \theta = \delta \tag{4.2}$$
$$\delta \bullet \delta = \delta \tag{4.3}$$
$$\delta \bullet \lambda = \lambda \bullet \delta = \lambda \text{ (or } \delta) \text{ for all } \lambda \in E \tag{4.4}$$

Then (Y , ● , θ) is an U- algebra.

Theorem(4.2): Let (E , * , θ) be an U- algebra. Let δ , η ∉ E and let Z= E ∪ { δ , η }.
A binary operation Θ be defined on Z as follows

$$\lambda \Theta \mu = \lambda * \mu ; \lambda, \mu \in E \tag{4.5}$$
$$\theta \Theta \delta = \delta \Theta \theta = \delta, \tag{4.6}$$
$$\theta \Theta \eta = \eta \Theta \theta = \eta, \tag{4.7}$$
$$\delta \Theta \delta = \delta , \eta \Theta \eta = \eta, \tag{4.8}$$
$$\delta \Theta \lambda = \lambda \Theta \delta = \lambda , \lambda \in E \tag{4.9}$$
$$\eta \Theta \lambda = \lambda \Theta \eta = \eta , \lambda \in E \tag{4.10}$$
$$\delta \Theta \eta = \eta \Theta \delta = \eta. \tag{4.11}$$

then (Z , Θ , θ) is an U- algebra.

Corollary (4.3): Two point extensions are also possible under following methods:

Method 1: The binary operation Θ_1 satisfies conditions (4.5) to (4.8). and

$$\delta \Theta_1 \lambda = \lambda \Theta_1 \delta = \delta, \tag{4.12}$$
$$\lambda \Theta_1 \eta = \eta \Theta_1 \lambda = \lambda, \tag{4.13}$$
$$\delta \Theta_1 \eta = \eta \Theta_1 \delta = \delta. \tag{4.14}$$

Method 2: The binary operation Θ_2 satisfies conditions (4.5) to (4.8) and

$$\delta \Theta_2 \lambda = \lambda \Theta_2 \delta = \delta, \tag{4.15}$$
$$\eta \Theta_2 \lambda = \lambda \Theta_2 \eta = \eta, \tag{4.16}$$
$$\eta \Theta_2 \delta = \delta \Theta_2 \eta = \eta. \tag{4.17}$$

Method 3: The binary operation Θ_3 satisfies conditions (4.5) to (4.8) ; (4.15) (4.16) and

$$\eta \Theta_3 \delta = \delta \Theta_3 \eta = \delta \tag{4.18}.$$

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