SOME FIXED POINT THEOREMS IN b-METRIC SPACES WITH MODIFIED CONTRACTIVE MAPPINGS

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Abstract: In this paper, we introduced a new concept of contractions which are denoted as $\prec$ -contractive mapping of sort-I and $\prec$ -contractive mapping of sort-II and we give some examples which are supporting to our main theorems.

Keywords: fixed point, b-metric space, Cauchy sequence, $\prec$ -contractive mapping of sort-I and $\prec$ -contractive mapping of sort-II.

1. Introduction:

The well-known concept of metric space was introduced by M. Frechet in 1906 as an extension of usual distance. In the theory of metric space, Banach’s contraction principle is one of the most celebrated fixed point theorem. It has been generalized in various directions. Many mathematicians studied a lot of interesting extensions and generalizations. It is important as a source of existence and uniqueness theorems in different branches of sciences. It states “Any contractive mapping on a complete metric space has a unique fixed point”. A mapping $T : X \to X$, where $(X, d)$ is called a contraction mapping if there exists a real number $\alpha < 1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \alpha d(x, y)$. If the metric space $(X, d)$ is complete, then $T$ has a unique fixed point.

In [12], Kannan proved the following result which gives the fixed point for discontinuous mapping: let $(X, d)$ be a complete metric space and $T : X \to X$ be a self map of $X$. Suppose that there exists $\beta \in [0,\frac{1}{2})$, such that $d(Tx, Ty) \leq \beta (d(x, Tx) + d(y, Ty))$ for all $x, y \in X$. Then $T$ has a unique fixed point in $X$.

Contraction mappings have been extended or generalized in several directions by various authors (see, for example, (5,12,17,21,23)). Not only contraction mappings but the concept of metric space is also extended in many ways in the literature (see for example,3,6,24,9). The concept of b-metric space was initiated by Bhaktin and Czerwik (6,7) as an extension of metric spaces by weakening the triangular inequality.

**Definition 1** ([3,6,7]). Let $X$ be a non-empty set. Then, a mapping $d : X \times X \to [0, +\infty)$ is called a b-metric if there exists a number $s \geq 1$ such that for all $x, y, z \in X$,

\[
\begin{align*}
(d1) & \quad d(x, y) = 0 \text{ if and only if } x = y; \\
(d2) & \quad d(x, y) = d(y, x); \\
(d3) & \quad d(x, z) \leq s(d(x, y) + d(y, z)).
\end{align*}
\]

Then triplet $(X, d, s)$ is called a b-metric space with index $s$. Clearly, every metric space is a b-metric space with $s = 1$, but the converse is not true in general. In fact, the class of b-metric spaces is larger than the class of metric spaces.

**Example 1:** Let $X = [0,2]$ and $d : X \times X \to [0, \infty)$ be defined by

\[
\begin{align*}
d(x, y) &= (x - y)^2, \quad x, y \in [0,1] \\
&\quad \frac{1}{x^2} - \frac{1}{y^2}, \quad x, y \in [1,2] \\
&\quad |x - y|, \quad \text{otherwise}
\end{align*}
\]

Show that $(x, d, s)$ is a b-metric space with index $s=2$.

1. $d(x, y) = 0 \iff (x - y)^2 = 0$ or $\frac{1}{x^2} - \frac{1}{y^2} = 0$ or $|x - y| = 0$.

   $\iff x = y \quad x = y \quad x = y$

2. $d(x, y) = (x - y)^2$ or $\frac{1}{x^2} - \frac{1}{y^2} = 0$ or $|x - y| = y - x$.

   $= (y - x)^2$ or $\frac{1}{y^2} - \frac{1}{x^2} = 0$ or $|y - x| = y - x$
3. Let \( x, y, z \in [0,1] \),
\[
d(x, z) = (x - z)^2 = [(x - y) + (y - z)]^2
\]
\[
\leq 2 [(x - y)^2 + (y - z)^2] \quad \text{(since } (a + b)^2 \leq 2(a^2 + b^2))
\]
\[
\leq 2 \{d(x, y) + d(y, z)\}
\]

II) Let \( x, y, z \in [1,2] \)
\[
d(x, y) = \left| \frac{1}{x^2} - \frac{1}{z^2} \right| = \left| \frac{1}{x^2} - \frac{1}{y^2} + \frac{1}{y^2} - \frac{1}{z^2} \right|
\]
\[
\leq \left| \frac{1}{x^2} - \frac{1}{y^2} \right| + \left| \frac{1}{y^2} - \frac{1}{z^2} \right| \quad \text{(since } |a+b| \leq |a| + |b|)
\]
\[
\leq d(x, y) + d(y, z)
\]

III) Otherwise, \( d(x, z) = |x - z| = |x - y + y - z|
\]
\[
\leq |x - y| + |y - z|
\]
\[
\leq d(x, y) + d(y, z)
\]
\[
\therefore (x, d, s) \text{ is a b-metric space with index } s = 2
\]

In [7], Banach’s contraction principle is proved in the framework of b-metric spaces. In 2013, Kir and Kiziltunc established the results in b-metric spaces, which generalized the Kannan and Chatterjea type mappings. In [1], the authors introduced the following result that improves Theorem 1 in [8].

**Theorem 1** ([1]). Let \((X, d)\) be a complete b-metric space with a constant \( s \geq 1 \).

If \( T : X \to X \) satisfies the inequality:
\[
d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 d(x, y) + d(Tx, Ty),
\]
where \( \lambda_1 \geq 0 \) for all \( i = 1, 2, 3, 4 \) and \( \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1 \) for \( s \in [1, 2] \) and \( 2/s < \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1 \) for \( s \in (2, +\infty) \); then, \( T \) has a unique fixed point.

In [5], the author introduced quasi-contraction mappings in b-metric spaces with some more restriction on values of \( q \) (see, for example [1,2,11,20,22]). More on b-metric spaces can be found in [13,14,15,16,18,19].

In the present work, we define a new class of functions. After that, we define some new contractive mappings which combine the terms \( d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) \) and \( d(Tx, Ty) \) and also will replace \( \frac{1}{s} \) with \( \alpha \), where \( \alpha \in \left[\frac{1}{s}, 1\right] \) by means of the member of a newly defined class. We also prove some fixed point theorems. To prove our theorems we need the following concepts and results from the literature.

**Definition 2** ([15]). Let \((X, d, s \geq 1)\) be a b-metric space. Then, a sequence \(\{x_n\}\) in \(X\) is called:

1. **Cauchy sequence** if for each \( \epsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon \) for all \( n, m \geq n_0 \).

2. **Convergent** if there exists \( l \in X \) such that for each \( \epsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that \( d(x_n, l) < \epsilon \) for all \( n \geq n_0 \). In this case, the sequence \(\{x_n\}\) is said to converge to \(l\).

**Definition 3** ([15]). A b-metric space \((X, d, s \geq 1)\) is said to be complete if every Cauchy sequence is convergent in it.

**Lemma 1** ([16]). Let \((X, d, s \geq 1)\) be a b-metric space and suppose that sequences \(\{x_n\}\) and \(\{y_n\}\) converge to \(x \) and \(y \in X\), respectively. Then,
\[
\frac{1}{s^2} d(x, y) \leq \lim_{n \to \infty} \inf d(x_n, y_n) \leq \lim_{n \to \infty} \sup d(x_n, y_n) \leq s^2 d(x, y)
\]

In particular if \( x = y \), then \( \lim_{n \to \infty} d(x_n, y_n) = 0 \)

Moreover, for any \( z \in X \), we have
\[
\frac{1}{s} d(x, z) \leq \lim_{n \to \infty} \inf d(x_n, z) \leq \lim_{n \to \infty} \sup d(x_n, z) \leq s d(x, z)
\]

**Lemma 2** ([18]): Every sequence \(\{x_n\}\) of elements from a b-metric space \((X, d, s \geq 1)\) having the property that there exists \( \lambda \in (0,1) \) such that \(d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})\) for every \( n \in \mathbb{N} \) is Cauchy.

**2.Fixed Point Theorems in b-Metric Spaces**

In this section, we first define a new class of functions, and then we define a new contractive mapping in b-metric spaces as follows.

**Definition 4:**
For any \( m \in \mathbb{N} \), we define \( \Xi_m \) to be the set of all functions \( \xi : [0, +\infty)^m \to [0, +\infty) \) such that

\( \xi(1) \xi(t_1, t_2, ..., t_m) > \max\{t_1, t_2, ..., t_m\} \) if \( t_1, t_2, ..., t_m \neq (0, 0, ..., 0) \);

\( \xi(1) \xi(t_1, t_2, ..., t_m) \) if \( \{t_i^{(n)}\}_{i=1}^m \neq \{t_1, t_2, ..., t_m\} \) for \( 1 \leq i \leq m \), are \( m \) sequences in \([0, +\infty) \) such that \( \lim_{n \to \infty} \sup t_i^{(n)} = t_i < +\infty \) for all \( i = 1 \) to \( m \), then
\[
\lim_{n \to \infty} \inf \left( \xi(t_1^{(n)}, t_2^{(n)}, ..., t_m^{(n)}) \right) \leq \xi(t_1, t_2, ..., t_m)
\]
Main Result-1:

**Definition 5:** Let \((X, d, \infty)\), where \(\infty \epsilon \left[\frac{1}{s}, 1\right] \), \(s \geq 1\) be a b-metric space. The mapping \(T: X \rightarrow X\) is said to be an \(\infty -contractive\) mapping of sort-1 if there exists \(\infty \epsilon \Xi\) s and

\[
d(Tx, Ty) \leq \infty \xi(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s})
\]

For all \(x, y \in X\).

Now the first result of this paper is as follows:

**Theorem 2:** Let \((X, d, \infty)\), where \(\infty \epsilon \left[\frac{1}{s}, 1\right] \), \(s \geq 1\) be a complete b-metric space and \(T: X \rightarrow X\) be an \(\infty -contractive\) mapping of sort-1, then \(T\) has a unique fixed point.

**Proof:** Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) in \(X\) as \(x_n = Tx_{n-1}\) for all \(n \geq 1\). Assume that any two consecutive terms of the sequence \(\{x_n\}\) are distinct; otherwise \(T\) has a fixed point.

First, we prove that \(\{x_n\}\) is a Cauchy sequence.

For this let \(n \in N\).

Consider, \(d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \infty \xi(d(x_{n-1}, x_n), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s}) < \infty \max\left\{d(x_{n-1}, x_n), d(x, x_{n+1}), d(x, x_{n+1})\right\} \frac{d(x_{n-1}, x_{n+1})}{2s} \leq \infty \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\right\} \frac{d(x_{n-1}, x_{n+1})}{2s}

Which implies that \(d(x_n, x_{n+1}) < \infty d(x_n, x_{n+1})\) for all \(n \geq 1\).

**Case-1:** If \(\frac{1}{s} \leq \infty < 1\), then by Lemma-2 in view of (3), \(\{x_n\}\) is a Cauchy sequence.

**Case-2:** If \(\infty = 1\), then by (3), the sequence \(\{d(x_n, x_{n+1})\}\) is monotonically decreasing and bounded below. Therefore, \(d(x_n, x_{n+1}) \rightarrow k\) for some \(k \geq 0\). Suppose that \(k > 0\); now taking limit inf in (2), we have \(k \leq \infty (k, k, k', k')\), where

\[
k' = \lim_{n \rightarrow \infty} \sup_{k < 1} \frac{d(x_{n-1}, x_{n+1})}{2} \leq \lim_{n \rightarrow \infty} \sup_{k < 1} \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} = k
\]

Now \(k \leq \infty (k, k, k')\) which contradicts the fact that \(k = k\).

Therefore, \(d(x_n, x_{n+1}) = 0\).

Suppose that \(\{x_n\}\) is not a Cauchy sequence; then there exists \(\epsilon > 0\) such that for any \(r \in N\), there exists \(m_r > n_r \geq r\) such that \(d(x_{m_r}, x_{n_r}) \geq \epsilon\).

Furthermore, assume that \(m_r\) is the smallest natural number greater than \(n_r\) such that (5) holds. Then

\[
\infty \leq d(x_{m_r}, x_{n_r}) \\
\leq d(x_{m_r}, x_{n_r}) + d(x_{n_r}, x_{n_r}) \\
\leq d(x_{m_r}, x_{n_r}) + d(x, x_{n_r}) + \epsilon \\
\leq d(x_{m_r}, x_{n_r}) + \epsilon
\]

Thus using (4) and taking limit, we get \(\lim_{r \rightarrow \infty} d(x_{m_r}, x_{n_r}) = \epsilon\).

Now, consider

\[
d(x_{m_r+1}, x_{n_r+1}) \leq \xi(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{n_r+1}), d(x_{n_r}, x_{n_r+1}), d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r+1}) \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r+1})}{2}
\]

Therefore, we have

\[
d(x_{m_r}, x_{n_r}) \leq d(x_{m_r}, x_{n_r+1}) + d(x_{n_r}, x_{n_r+1}) + d(x_{n_r}, x_{n_r}) \leq d(x_{m_r}, x_{n_r+1}) + d(x_{n_r}, x_{n_r+1}) + \xi(d(x_{m_r}, x_{n_r}), d(x_{m_r}, x_{n_r+1}), d(x_{n_r}, x_{n_r+1}), d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r+1}) \frac{d(x_{m_r}, x_{n_r+1}) + d(x_{m_r+1}, x_{n_r+1})}{2}
\]

Thus by taking limit inf on both sides and also using (4) and (6), we get

\[
\epsilon \leq 0 + 0 + \xi(\epsilon, \epsilon, \epsilon', \epsilon)\)

\[
\epsilon' = \lim_{r \rightarrow \infty} \sup_{k < 1} \frac{d(x_{m_r}, x_{n_r}) + d(x_{m_r}, x_{n_r}))}{2} \leq \lim_{r \rightarrow \infty} \sup_{k < 1} \frac{d(x_{m_r}, x_{n_r}) + d(x_{m_r}, x_{n_r}) + d(x_{m_r}, x_{n_r})}{2} = \epsilon
\]

Thus \(\epsilon \leq \xi(\epsilon, \epsilon, \epsilon', \epsilon') < \max\{\epsilon, 0, 0, \epsilon\}' = \epsilon\), which is a contradiction. Thus \(\{x_n\}\) is a Cauchy sequence in \((X, d, \infty)\), where \(\infty \epsilon \left[\frac{1}{s}, 1\right]\).

Now \((X, d, \infty)\) is a complete b-metric space. Therefore, there exists \(x \in X\) such that \(x_n \rightarrow x\).

Now, consider

\[
d(Tx_n, Tx) \leq \infty \xi(d(x_n, x), d(x_n, Tx_n), d(x, Tx), d(x, Tx_n) + d(x, Tx_n) \frac{d(x_n, Tx_n) + d(x, Tx_n)}{2s})
\]

Which implies that
\[ d(x_{n+1}, Tx) \leq \xi \left( d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x, Tx) + d(x, x_{n+1})}{2} \right) \]

Taking \( \lim \inf \) on both sides and using Lemma-1, we get
\[ \frac{1}{s} d(x, Tx) \leq \xi \left( 0, 0, d(x, Tx), l \right) \]
i.e., \( d(x, Tx) \leq \xi \left( 0, 0, d(x, Tx), l \right) \)
where \( l = \lim \sup_{n \to \infty} \frac{d(x_n, Tx) + d(x_n, x_{n+1})}{2s} \leq \lim_{n \to \infty} \frac{d(x, Tx) + d(x, x_{n+1})}{2s} \)
Thus \( d(x, Tx) \leq \xi \left( 0, 0, d(x, Tx), l \right) \) and \( \max \{0, d(x, Tx), l\} = d(x, Tx) \)
Which is a contradiction. Therefore, \( Tx = x \)

Let \( Ty = y \) for some \( y \in X \) and suppose that \( x \neq y \), then consider
\[ d(x, y) = d(Tx, Ty) \leq \xi \left( d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right) \]
\[ \leq \xi \left( d(x, y), 0, 0, \frac{d(x, y)}{s} \right) \]
\[ < \xi \max \left\{ d(x, y), 0, 0, \frac{d(x, y)}{s} \right\} \]
\[ = \xi d(x, y), \text{ which is a contradiction. Therefore } x = y. \]

Now, the following result is a consequence of Theorem-2.

**Corollary-1:** Let \((X, d, s)\) where \( s \geq 1 \) be a complete \( b \)-metric space and \( T : X \to X \) be a mapping such that there exists \( q \in \left[ 0, \frac{1}{s} \right] \) and \( d(Tx, Ty) \leq q \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \} \) \quad \text{for all } x, y \in X. \)
Then \( T \) has a unique fixed point.

**Proof:** Let \( \xi \in \rho_s \) be defined by \( \xi(t_1, t_2, t_3, t_4) = qs \max \{ t_1, t_2, t_3, t_4 \} \). Then following Theorem-2, \( T \) has a unique fixed point.

In the following example, we see that conditions of theorem 2 are satisfied, but corollary 1 is not applicable.

**Example 1:** Let \( X = \left\{ \frac{1}{\sqrt{n}} \mid n \in \mathbb{N} \right\} \cup \{0\} \). Define \( d : X \times X \to [0, \infty) \) by \( d(x, y) = \left| x - y \right|^2 \) for all \( x, y \in X \). Then \( d \) is a \( b \)-metric on \( X \) with \( s=2 \).

Solution: Define \( T : x \to x \) by \( T(x) = \frac{1}{\sqrt{2n+1}} \) for all \( n \in \mathbb{N} \) and \( T(0)=0 \).

Define \( \xi(t_1, t_2, t_3, t_4) = \begin{cases} \frac{max(t_1, t_2, t_3, t_4)}{1+t_4}, & \text{if } t_4 > 0 \\ \frac{max(t_2, t_3, t_4)}{1+t_4}, & \text{otherwise} \end{cases} \)

Now, for all \( x, y \in X \), (1) is satisfied, and thus the conditions of Theorem 2 are satisfied. However, we see that if (7) is satisfied for all \( x, y \in X \), we have \( d(Tx, Ty) \leq qN(X, Y) \)
For all \( x, y \in X \), where \( N(x, y) = max \{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx) \} \)
So, in particular, we have
\[ d \left( \frac{1}{\sqrt{2n+1}}, \frac{1}{\sqrt{2m+1}} \right) \leq q N \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}} \right) \]
\[ \text{for all } m, n \in \mathbb{N} \text{, } m \neq n. \]
\[ \text{i.e., } \frac{\left| \frac{1}{\sqrt{2n+1}} - \frac{1}{\sqrt{2m+1}} \right|^2}{\left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}} \right)^2} \leq 2q \text{ for all } m, n \in \mathbb{N} \text{, } m \neq n. \]

Now, taking \( \lim \) we get \( 2q \geq 1 \), which is a contradiction.

Thus, corollary 1 is not applicable for this example.

**Corollary 2:** Let \((X, d, s \geq 1)\) be a complete \( b \)-metric space and \( T : X \to x \) be a mapping such that
\[ d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 [d(x, Ty) + d(y, Tx)] \]
\[ \text{for all } x, y \in X, \text{where } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq \frac{1}{2}, \lambda_i \geq 0 \text{ for all } i=1 \text{ to } 4. \]
Then \( T \) has a unique fixed point.

**1.2 Main result 2**

Now, we define another contractive mapping in \( b \)-metric space

**Definition 6:** Let \((X, d, a)\), where \( a \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) and \( s \geq 1 \) be a \( b \)-metric space.

The mapping \( T : X \to X \) is said to be an \( \xi - \text{Contraction mapping of sort-II} \) if there exists \( \xi \in \Xi_s \) and \( d(Tx, Ty) \leq a d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \)
\[ \text{for all } x, y \in X. \]

The proof of our next result proceeds in a similar manner as the proof of Theorem 2.

**Theorem 3:** Let \((X, d, a)\), where \( a \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) and \( s \geq 1 \) be a complete \( b \)-metric space and \( T : X \to x \) be an \( \xi - \text{Contraction mapping of sort-II} \) then \( T \) has a unique fixed point.

**Corollary 3:** Let \((X, d, s \geq 1)\) be a complete \( b \)-metric space and \( T : X \to x \) be a mapping such that there exists \( q \in \left[ 0, \frac{1}{s} \right] \) and \( d(Tx, Ty) \leq q \max \{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \}
\[ \text{for all } x, y \in X. \]

Then \( T \) has a unique fixed point.

**Proof:** Let \( \xi \in \Xi_s \) be defined by \( \xi(t_1, t_2, t_3, t_4, t_5) = qs \max \{ t_1, t_2, t_3, t_4, t_5 \} \). Then, by theorem 3, \( T \) has a Unique fixed point.
Corollary 4: Let \((x,d,s \geq 1)\) be a complete b-metric space and \(T : X \rightarrow X\) be a mapping such that
\[
d(Tx, Ty) \leq \alpha_1 d(x,y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Tx) + \alpha_5 d(Tx, y)\]
for all \(x, y \in X\), where \(\alpha_1 + \alpha_2 + \alpha_3 + \delta \alpha_4 + \alpha_5 < \frac{1}{s}\) and \(\alpha_i \geq 0\) for all \(i = 1 \text{to} 5\). Then, \(T\) has a unique fixed point.

Proof: Let \(\xi \in \Xi\) be defined by \(\xi(t_1, t_2, t_3, t_4, t_5) = s(\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 + \delta \alpha_4 t_4 + \alpha_5 t_5)\). Then by theorem 3, \(T\) has a unique fixed point.

Conclusions: In this paper, we have defined a new class of functions, and with the help of this class of functions, we defined some new contractive mappings in b-metric spaces. Furthermore, we proved some fixed point results for these contractive mappings like \(\alpha\)-contraction of sort -I and \(\alpha\)-contraction of sort -II.

References: