"ON CHARACTERISATION OF FIXED POINT THEOREMS IN CONVEX METRIC SPACES"

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Abstract- The main purpose here is to show equivalence between Fixed point theorem and convex metric space. It ply crucial role in mathematical theory with applications in various branches of science. Banach contraction principle was introduced by S. Sedghi(9) and some common fixed point and S-metric space results for self-mappings on vector valued complete S-metric space. We derive some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. We also investigate some common fixed point theorems for a Banach operator pair under certain generalized contractions on a convex complete metric space derived by S. Sedghi(9). It has important role in fixed point theory and became very famous due to iterations used in the theorem. The evaluation of fixed points of mappings satisfying many contractive conditions is at the center of research work and several vital results have been established by T. Takahashi(5).

Keywords: Fixed Points, metric space, convex metric space, mapping, S-metric space.

1. INTRODUCTION

Takahashi introduced the notion of convexity in metric spaces and studied some fixed point theorems for non-expansive mappings in such spaces. A convex metric space is a generalized space for every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Beg, Beg and Abbas, Chang, Kim studied fixed point theorems in convex metric spaces. Karapinar proved that for a closed convex subset C of a cone Banach space X with the norm ||x||_p = d(x, 0), if a mapping T : C → C satisfies the condition

\[ d(x, Tx) + d(y, Ty) \leq qd(x, y) \quad \text{\ldots (1.1)} \]

for all x, y ∈ C, where 2 ≤ q < 4, then T has at least one fixed point. Letting x = y in the above inequality, it is easy to see that T is an identity mapping. The above result is improved and extended to a convex complete metric space.

2. DEFINITION

(i) Convex structure
Let (X, d) be a metric space and I = [0, 1]. A mapping W : X × X × I → X is said to be a convex structure on X if for each (x, y, λ) ∈ X × X × I and u ∈ X,

\[ d_W (x, y, λ) \leq λd(u, x) + (1 - λ)d(u, y) \]

A metric space (X, d) together with a convex structure W is called a convex metric space, which is denoted by (X, d, W).

(ii) Convex
Let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be convex if W(x, y, λ) ∈ C whenever (x, y, λ) ∈ C × C × I.

(iii) Self mapping
Let (X, d, W) be a convex metric space and C be a convex subset of X. A self-mapping f on C has a property (I) if f(W(x, y, λ)) = W(f(x), f(y), λ) for each x, y ∈ C and λ ∈ I.

3. THEOREM

Let C be a closed and convex subset of a cone Banach space X with the norm ||x||_p = d(x, 0), and T : C → C be a mapping which satisfies the condition

\[ d(x, Tx) + d(y, Ty) \leq qd(x, y) \quad \text{\ldots (1.2)} \]

for all x, y ∈ C, where 2 ≤ q < 4. Then, T has at least one fixed point.

4. THEOREM

Let C be a closed and convex subset of a cone Banach space X with the norm ||x||_p = d(x, 0), and T : C → C be a mapping which satisfies the condition

\[ d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y) \quad \text{\ldots (1.3)} \]
for all $x, y \in C$, where $2 \leq r < 5$. Then, $T$ has at least one fixed point. Before proving the main theorem, we derive the following Theorem.

5. THEOREM

Let $(X, d, W)$ be a convex metric space, then the following statements hold:

(i) $d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))$ for all $(x, y, \lambda) \in X \times X \times I$.

(ii) $d(x, W(x, y, \frac{1}{2})) = d(y, W(x, y, \frac{1}{2})) = \frac{1}{2} d(x, y)$ for all $x, y \in X$.

Proof

(i) For any $(x, y, \lambda) \in X \times X \times I$, we have

\[ d(x, y) \leq d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(x, y) = d(x, y). \]

Therefore, $d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))$ holds.

(ii) Let $x, y \in X$. By the definition of $W$ and using (i), we have

\[ d\left(x, W\left(x, y, \frac{1}{2}\right)\right) \leq \frac{1}{2} d(x, y) \]

Therefore, we obtain

\[ \frac{1}{2} d\left(x, W\left(x, y, \frac{1}{2}\right)\right) \leq \frac{1}{2} d\left(y, W\left(x, y, \frac{1}{2}\right)\right) \quad \text{... (1.4)} \]

Similarly, we have

\[ \frac{1}{2} d\left(y, W\left(x, y, \frac{1}{2}\right)\right) \leq \frac{1}{2} d\left(x, W\left(x, y, \frac{1}{2}\right)\right) \quad \text{... (1.5)} \]

Therefore, $d(x, W(x, y, \frac{1}{2})) = d(y, W(x, y, \frac{1}{2}))$. Now, from (i), we obtain

\[ d\left(x, W\left(x, y, \frac{1}{2}\right)\right) = d\left(y, W\left(x, y, \frac{1}{2}\right)\right) = \frac{1}{2} d(x, y) \quad \text{... (1.6)} \]

for all $x, y \in C$, and the proof of the theorem is complete. We derive theorem which extend the result due to V. Runde\(^{[2]}\).

6. THEOREM

Let $C$ be a nonempty closed convex subset of a convex complete metric space $(X, d, W)$ and $f$ be a self-mapping of $C$. If there exist $a, b, c, k$ such that

\[ 2b - |c| \leq k < 2(a + b + c) - |c|, \]

\[ ad x, f(x) + bd y, f(y) + cd f(x), f(y) \leq kd(x, y) \]

for all $x, y \in C$, then $f$ has at least one fixed point.

Proof

Let us choose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

\[ x_n = W\left(x_{n-1}, f\left(x_{n-1}, \frac{1}{2}\right)\right) \quad n = 1, \ldots \quad \text{... (1.7)} \]

As $C$ is convex, $x_n \in C$ for all $n \in N$. By using hypothesis of Theorem (4), we have

\[ d(x_n, f(x_n)) = 2d(x_n, x_{n+1}) \]

\[ d(x_n, f(x_n)) = d(x_n, x_{n+1}) \quad \text{... (1.8)} \]

for all $n \in N$. Now, by substituting $x$ with $x_n$ and $y$ with $x_{n+1}$ in (1.8), we get

\[ ad(x_n, f(x_n)) + bd(x_n, f(x_n)) + cd(f(x_n), f(x_n)) \leq kd(x_n, x_{n+1}) \]

for all $n \in N$. Therefore, combining (1.7) and (1.8), it follows that

\[ 2ad(x_n, x_{n+1}) + 2bd(x_n, x_{n+1}) + cd(f(x_n), f(x_n)) \leq kd(x_n, x_{n+1}) \]

for all $n \in N$. Let $c$ be a nonnegative number. We obtain

\[ 2ad(x_n, x_{n+1}) - cd(x_n, x_{n+1}) \leq cd(f(x_n), f(x_n-1)) \]

for all $n \in N$. Similarly, for the case $c < 0$, we have

\[ 2ad(x_n, x_{n+1}) + cd(x_n, x_{n+1}) \leq cd(f(x_n), f(x_n-1)) \]

for all $n \in N$. Therefore, for each case we have

\[ 2ad(x_n, x_{n+1}) - |c|d(x_n, x_{n+1}) \leq cd(f(x_n), f(x_n-1)) \quad \text{... (1.9)} \]

for all $n \in N$. Now, again combining (1.7) and (1.9), it follows that

\[ 2ad(x_n, x_{n+1}) + 2bd(x_n, x_{n+1}) - |c|d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1}) \]

for all $n \in N$. This implies

\[ d(x_n, x_{n+1}) \leq \frac{k - 2b + |c|}{2(a + c)} d(x_n, x_{n+1}) \quad \text{... (1.10)} \]
for all \( n \in \mathbb{N} \). From (1.9) \( k - 2b + |c| \in [0, 1) \), and hence, \( \{x_n\}_{n=1}^{\infty} \) is a contraction sequence in \( C \). Therefore, it is a Cauchy sequence. Since \( C \) is a closed subset of a complete space, there exists \( v \in C \) such that \( \lim_{n \to \infty} x_n = v \). Therefore, the triangle inequality and it imply \( \lim_{n \to \infty} f(x_n) = v \). Now, by substituting \( x \) with \( v \) and \( y \) with \( x \) in (1.10), we obtain 
\[
\text{adv}, f(v) + bd(x_n, f(x_n)) + cdf(v, f(x_n)) \leq kdv(\sqrt[n]{x_n}, x_n) \tag{1.11}
\]
for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) in the inequality (1.11), it follows that 
\[
(a + c)dv, f(v) \leq 0 \tag{1.12}
\]
Since \( a + c \) is positive from (1.12), it follows that \( dv, f(v) = 0 \). Therefore, \( f(v) = v \) and the proof of the theorem is complete. We derive corollary, which generalises the results due to F. Karapinar.

7. THEOREM
Let \((X, d, W)\) be a convex complete metric space and \( C \) be a nonempty subset of \( X \). Suppose that \( f, g \) are self-mappings of \( C \), and there exist \( a, b, c, k \) such that
\[
2b - |c| \leq k < 2(a + b + c) - |c|.
\]
\[
Ad(g(x), f(x)) + bd(y, f(y)) + cdf(x, f(y)) \leq kdv(x, y)
\]
for all \( x, y \in C \). If \((f, g)\) is a Banach operator pair, \( g \) has the property \((I)\) and \( F(g) \) is a nonempty closed subset of \( C \), then \( F(f, g) \) is nonempty.

Proof
By an application of (1.13), we obtain
\[
Ad(x, f(x)) + bd(y, f(y)) + cdf(x, f(y)) \leq kdv(x, y)
\]
for all \( x, y \in F(g) \). \( F(g) \) is convex because \( g \) has the property \((I)\). It follows from Theorem (6) that \( F(f, g) \) is nonempty.

8. THEOREM
Let \((X, d, W)\) be a convex complete metric space and \( C \) be a nonempty subset of \( X \). Suppose that \( f, g \) are self-mappings of \( C \), \( F(g) \) is a nonempty closed subset of \( C \), and there exist \( a, b, c, k \) such that
\[
2b - |c| \leq k < 2(a + b + c) - |c|.
\]
\[
Ad(g(x), g(f(x))) + bd(y, g(f(y))) + cdf(g(x), g(f(y))) \leq kdv(g(x), g(y)) \tag{1.14}
\]
for all \( x, y \in C \). If \((f, g)\) is a Banach operator pair and \( g \) has the property \((I)\), then \( F(f, g) \) is nonempty.

Proof
Since \((f, g)\) is a Banach operator pair from (1.14), we have
\[
adx, f(x) + bdy, f(y) + cdf(x, f(y)) \leq kdv(x, y) \tag{1.15}
\]
for all \( x, y \in F(g) \). Because \( g \) has the property \((I)\) and \( F(g) \) is closed, Theorem (8) suggests that \( F(f, g) \) is nonempty. Hence, the theorem is proved.

REFERENCES: