Mathematical modeling for finding the thermal conductivity of solid materials

Farhan Babu[1], Akhilesh Lodwal[2]

[1]PG Scholar, [2]Assistant Professor
Mechanical Engineering Department
Devi AhilyaVishwavidyalaya, Indore, India

***Abstract* - Thermal conductivity refers to the ability of materials to conduct heat. It plays crucial roles in the design of engineering systems where temperature and thermal stress are major concerns. Typical methods of thermal conductivity measurement can be categorized as either steady-state or non-steady state. The article deals with transient heat conduction through the solid spherical shape when it is immersed into the pool of hot water. The design of experimental set-up is purely based on natural convection. So, various dimensionless numbers Pr, Gr, Ra, Nu have been studied to find the natural heat transfer coefficient (h). During transient heat conduction process through the material some part of heat stores in the material and temperatures-time data is recorded by using the data acquisition system. Classical lumped model is valid for lower Biot number. So, an effort has been done to implement higher Biot number. Based on analysis, modeling, temperature variation with time and geometry of material, solution is obtained to find the thermal conductivity of materials.**

***Keywords*: Transient heat conduction, Dimensionless numbers, Pr, Gr, Ra and Nu, Natural heat transfer coefficient (h), Biot number (Bi), Fourier number (Fo) and thermal conductivity (km)**

I. INTRODUCTION

Thermal conductivity plays crucial roles in the design of engineering systems where temperature and thermal stress are of concerns. It is an intensive physical property of a material that relates the heat flow through the material per unit area to temperature gradient across the material. The thermal conductivity of a material is basically a measure of its ability to conduct heat. In a wide variety of applications ranging from building insulation to electronics, it is important to determine the thermal conductivity of materials.Extensive efforts have been made since 1950s for the characterization of thermal conductivity. Typical methods of thermal conductivity measurement can be categorized as either steady-state or non-steady state. In steady-state techniques, equilibrium heat flux and temperature gradient are measured. Transient techniques usually measure time-dependent energy dissipation process of a sample.

Major drawbacks of this technique include: (1) the testing sample should be relatively large, in centimeter scale or even larger. (2) The test usually suffers from a long waiting time, up to a few hours, to reach steady state.The biggest challenge in the steady state technique is to accurately determine the heat flow through sample. However, if one has a standard material whose thermal conductivity is known, the comparative cut bar technique can be applied and the direct measurement of heat flow is unnecessary.

The method is asked to investigate involves the transient heating of spherical shaped samples by the natural convection means. The temperature of water is increased by supplying heat to the water bath. The hot water rises up and cold water goes down as density decreases with increase in temperature for fluids and cycle begins to circulate the water. After maintaining the particular temperature of the fluid, a solid spherical shape is initially at ambient temperature, immersed into the hot water, until the whole of the shape temperature rises up with time continuously and it try to reaches the equilibrium/ steady state.

Thus following are the objectives of this project work.

❖ To find the dimensionless numbers like Prandtl number (Pr),) Rayleigh number (Ra) and Grashof number (Gr) and natural heat transfer coefficient (hc) by using experimental data and correlation of Nusselt number (Nu) for the spherical shape.

❖ To find the Fourier number (Fo) and Biot number (Bi) by using experimental data and Heisler chart for spherical shape.

❖ To find the thermal conductivity and validate the results of two different materials (Km) by using the analytical technique and mathematical modeling.

❖ To find the heat transfer due to conduction, convection, radiation and total amount of heat utilize during the various processes.

❖ To plot thermal conductivity v/s temperature of two different materials at various temperatures.

II. LITERATURE REVIEW

The thermal conductivity of materials widely used for their applications in various areas of Refrigeration and Air conditioning, Automobiles, Industries and Engineering etc.. In this chapter, literatures on thermal conductivity of solid materials and processes to find the thermal conductivity are discussed in details. The earliest work on Biot number were found.

Su, G., Tan, Z., & Su, J. (2007), An improved lumped models for transient heat conduction in a slab with temperature dependent thermal conductivity: where, they studied both cooling and heating processes and presented solutions as a function of the Biot number and a dimensionless parameter representing the heating or cooling process. Prasad, D. (2013). Analysis of transient heat conduction in different geometries by polynomial approximation method: where, he shows heat transfer generally takes place by three modes such as conduction, convection and radiation. Prakash, A., Mahmood, S., (2013). Modified lumped model for transient heat conduction spherical shape: where**,** unsteady or transient heat conduction state implies a change with time, usually only of the temperature. In general, the flow of heat takes place in different spatial coordinates. In some cases the analysis is done by considering the variation of temperature in one-dimension. At a spherical geometry to have one-dimensional analysis a uniform condition is applied to each concentric surface which bounds the region. Will, J. B., Kruyt, N. P., &Venner, C. H. (2017). An experimental study of force convective heat transfer from smooth, solid spheres: where, they use the rate of heat transfer due to natural convection is estimated for the spherical shape, based on a correlation for the Nusselt number as a function of the Rayleigh number (Ra) and the Prandtl number (Pr)

If, $Ra \le 10^{11}$; $Pr \ge 0.7$,

For the spherical shape, Nusselt number is given below,

$$\text{Nusselt no. (Nu)} = 2 + \frac{0.589 x Ra^{0.25}}{[1+\{\frac{0.469}{Pr}\}^{0.563}]^{0.44}}$$

Where:

$$Gr = \{g\beta(T_S - T_\infty)D^3\}/\nu^2$$

$$and \; Ra = Gr * Pr$$

g is the gravitational acceleration, β is the thermal expansion coefficient at constant pressure of the water and ν is the kinematic viscosity of the water and Pr is the prandtl number of water at given temperature.

III. METHODOLOGY

When, the temperature of water increases, the hot water rises up and cold water goes down as density decreases with increase in temperature for fluids and cycle begins to circulate the water. A solid shape is initially at ambient temperature, immersed into the hot water until the whole of the shape temperature rises up with time continuously and it try to reaches the equilibrium/ steady state. There is a convective heat transfer at the surface of the shape.There is conductive heat transfer inside of the shape due to the conduction of heat.

In case of natural convection, Nusselt number (Nu) is a function of Grashof number (Gr) and Prandtl number (Pr).

Natural convection

Nusselt number (Nu) = function [Gr, Pr]

By using the density, kinematic, dynamic viscosity and thermal conductivity of water at particular temperature, Prandtl number (Pr) can be found. It is a dimensionless number, named after the German physicist Ludwig Prandtl, is defined as the ratio of momentum diffusivity to thermal diffusivity.

$$\text{Prandtl number (Pr)} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\mu/\rho}{k/c_p\rho} = \frac{\mu c_p}{k}$$

where :

ν = momentum diffusivity (kinematic viscosity)

k = thermal conductivity of water at temperature

c_p = specific heat of water

This motion is caused by the buoyant force. The major force that resists the motion is the viscous force. The Grashof number (Gr) is a way to quantify the opposing forces.

The Grashof number (Gr) is defined as:

$$\text{Grashof number} = \frac{\text{Buoyancy force}}{\text{Viscous force}}$$

$$\text{Grashof number (Gr)} = \frac{D^3 \rho^2 g \Delta T \beta}{\mu^2}$$

D = Dimeter of the shape
ρ = Density of water
g = Gravitational acceleration
T_s = Temperature at the surface
T_i = Initial temperature
β = Coefficint of thermal expension
μ = Dynamic viscosity

The Rayleigh number (Ra) for a fluid is a dimensionless number associated with buoyancy-driven flow, also known as free convection or natural convection.

$$\text{Ra} = \text{Gr} * \text{Pr}$$

If, $\text{Ra} \leq 10^{11}$; $\text{Pr} \geq 0.7$,
For the spherical shape, Nusselt number is given,

$$\text{Nusselt number (Nu)} = 2 + \frac{0.589 x \text{Ra}^{0.25}}{[1+\{\frac{0.469}{\text{Pr}}\}^{0.563}]^{0.44}}$$

As, we know Nusselt number (Nu) is the ratio of convective heat transfer to conductive heat transfer across the surface boundary.

$$\textit{Nusselt} \text{ number (Nu)} = \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = \frac{h_c}{k_w / D} = \frac{h_c D}{k_w}$$

$$\text{Natural heat transfer coefficient } (h_c) = \text{Nu} * \left(\frac{k_w}{D}\right)$$

Where:
k_w is thermal conductivity of water at particular temperature
D is the diameter of the shape

$$\textit{Fourier} \text{ number (Fo)} = \frac{\alpha t}{D^2} = \frac{\text{The rate at which heat is conducted across thelength of the body}}{\text{The rate at which heat is stored in the body}}$$

α is thermal diffusivity of material
D is diameter of the body

$$\alpha = \frac{k}{\rho c_p}$$

k *is* conductivity of the body
ρ is density of the body

Initial temperature of the shape (T_i), *Temperature* of the hot water (T_0) and Temperature of the shape at step change (T_∞)
Dimensionless temperature or

$$\text{Temperature profile } (\theta_0) = \left[\frac{T_0 - T_\infty}{T_i - T_\infty}\right]$$

Heisler chart for the spherical shape

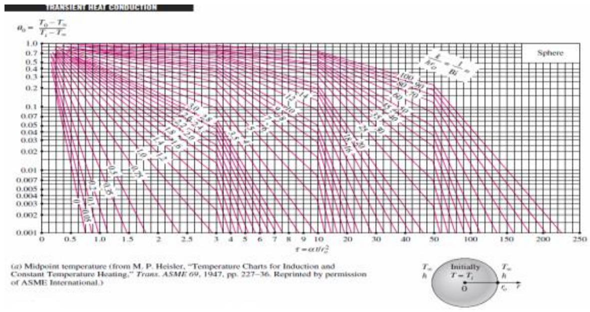


Figure 3.1: Heisler chart for the spherical shape

The heat flow experiences two resistances, the first within the solid metal (which is influenced by both the size and composition of the sphere), and the second at the surface of the sphere. The conductive component is measured under the same conditions as the heat convection.

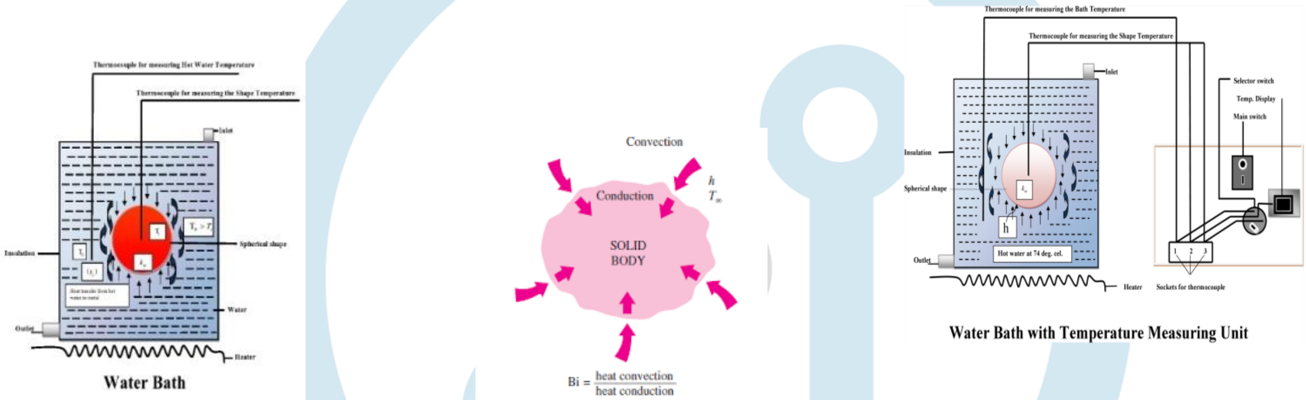


Figure 3.2: Schematic diagram of constant temperature water bath with solid spherical shape

$$\text{Biot number } (B_i) = \frac{\text{Conductive heat transfer resistance within the body}}{\text{Convective heat transfer resistance across the surface of the body}}$$

$$= \frac{L_c/k_m}{1/h_c} = \frac{h_c L_c}{k_m}$$

$$\text{Biot number } (B_i) = \frac{h_c L_c}{k_m}$$

$$\textit{So, Thermal conductivity} \text{ of the materail } [k_m] = h_c * \left[\frac{B_i}{L_c}\right]$$

Where:

h_c is heat transfer coefficient

L_c is the characteristic length of the shape

k_m is the thermal conductivity of the shape of the material

IV. EXPERIMENTAL SET-UP DEVELOPMENT

The objectives of the present study mentioned in the previous chapter, are mainly attained by specifically designed experimental set-up to perform temperature measurements during transient heat conduction. The current chapter provides the details of experimental set-up, technique used for the temperature measurements in the present study. This chapter is organized as per the following subsections: (i) Experimental set-up (ii) Technique for the temperature measurements.

Experimental set up used for this work is shown figure (Figure 3.1). The experimental set up comprises of a constant temperature water bath with inbuilt heater, a temperature measuring unit, data logging software and a Computer.

Figure 4.1: Experimental set-up

Solid spherical shapes

Two identical solid spherical shapes of different materials (Stainless steel and Brass) have been used in this experiment for their analysis. Each shape has been fitted with a thermocouple (electrical temperature sensor) that is located at the precise centre of the sphere.

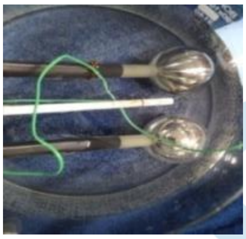


Figure 4.2: Spherical shape of Steel and Brass materials

EXPERIMENTAL PROCEDURE

After taking measurement of initial teperature of the shape, the solid shape is immersed in a water bath and measure the temperatures during transient heat conduction and it will also be rocorded by using data acquisition system. Before, immersing the solid shape into the water bath, its temperature has to be established at desired temperature. After, the bath emperature is stabilized. The solid shape is immersed into the water bath. First, look the solid to fish-eye on the frame. Insure, about data recording. Then, lower the solid into the water bath until completely submerged. Don't drop the solid in the bath, but try to get it immersed quickely. Time is noted down at the instant, the solid is immersed.

When, the solid has reached the temperature (step change) lower than the bath temperature and it tries to achieve steady state. Remove the solid and hang it back over the plastic tub so that it can be cool to room temperature. Make sure that the data are saved on the hard disk for the backup.

Repeat the same procedre to the other material.

V. EXPERIMENTAL DATA IMPLEMENTATION AND ANALYSIS

Table 5.1

Properties of water at 60, 70 and 80°C

S.N.	Properties of water	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	Density of water (ρ)	985.46	979.77	974.00
2.	Kinematic viscosity of water (ν)	$0.478x10^{-6}$	$0.421x10^{-6}$	$0.364x10^{-6}$
3.	Dynamic viscosity of water (μ)	$471x10^{-6}$	$412x10^{-6}$	$354x10^{-6}$
4.	Thermal conductivity of water (k_m)	0.651	0.660	0.668
5.	Prandtl number (Pr)	3.020	2.620	2.220

Table 5.2

Experimental Data – Steel material

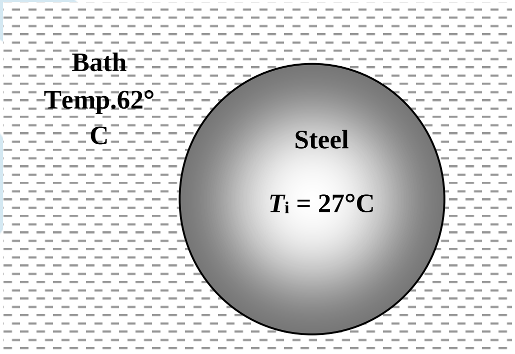
S.N.	Particulars	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	Diameter of spherical shape (mm)	45	45	45
2.	Characteristic length (mm)	7.5	7.5	7.5
3.	Material of spherical shape	Steel	Steel	Steel
4.	Bath temperature	62	74	82
5.	Initial temperature of shape	27	24	24
6.	Measured temperature (Step change)	60	70	80
7.	Time (during transient heat conduction)	180	200	220

Analysis : STEEL MATERIAL (Diameter 45 mm)

"*Case* – 1"

Material – Steel

Shape - Spherical



Initial shape temperature (T_i) = 27°C

Water bath temperature (T_∞)= 62°*C*

Measured temperature (T_0)= 60°C

at time (t) = 180 seconds

Pr*operties* of water at 60°C

Density (ρ) = 985.46 kg/m^3

Kinematic viscosity (ν) = 0.478x10^{-6} m^2/s

Dynamic viscosity (μ) = 471x10^{-6} m^2/*s*

Thermal conductivity (k) = 0.651 W/m°C

Prandtl number (Pr) = 3.02

β = Volumetric expension of the material

$$\beta = \frac{1}{T_f} = \frac{1}{(T_i + T_\infty)/2}$$

Using correlation for free convection

$$\text{Grashof number (Gr)} = \frac{D^3 \rho^2 g \Delta T \beta}{\mu^2}$$

Ra = Gr.Pr

If, Ra $\leq 10^{11}$; Pr ≥ 0.7

By calculation;

Gr =2.97x10^9

Pr = 3.02

So, Ra = 8.97x10^9

$$\text{Nusselt no. (Nu)} = 2 + \frac{0.589 \text{x} Ra^{0.25}}{[1 + \{\frac{0.469}{Pr}\}^{0.563}]^{0.44}}$$

and Nusselt no. (Nu) = 135

$$\text{Natural heat transfer coefficient (h)} = \frac{Nu}{D} k_w = \frac{135}{45\text{x}10^{-3}} * 0.651 = 1953 \ \frac{W}{m^2\ °C} \text{.................................(1)}$$

Dimensionless temperature $(\theta_0) = \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.0883$

Fourier number $(F_o) = \frac{\alpha}{D^2} t = 3.3$

From Heisler chart

Biot number (Bi)= 0.27

Biot number (Bi) $= \frac{h x L_c}{k_m}$

Thermal conductivity of material $(k_m) = \frac{L_c}{Bi} h$

$$(k_m) = \frac{7.5 x 10^{-3}}{0.27} h = 0.0277xh \text{...(2)}$$

From equation ......(1)

$$(k_m) = 54.00 \ \frac{W}{m\ ^{\circ}C}$$

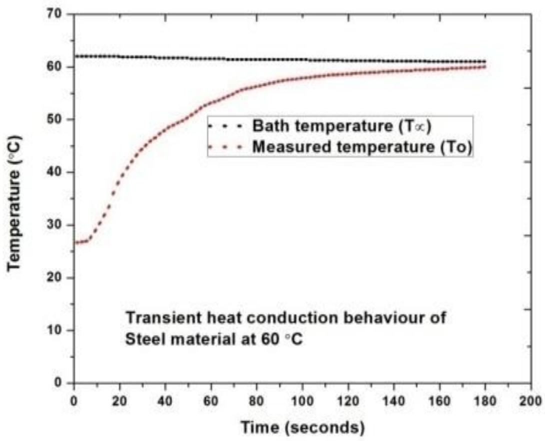


Figure 5.1: Transient heat conduction behavior of steel material at 60°C

Table 5.3

Properties of water at 60, 70 and 80°C

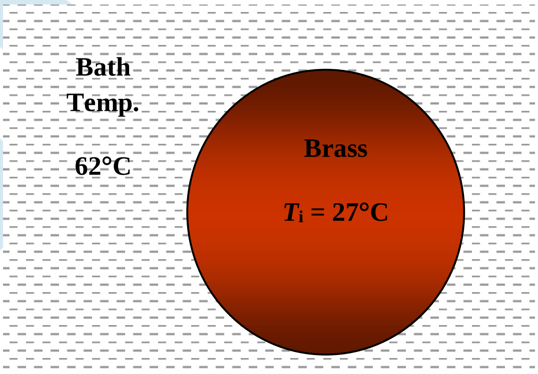
S.N.	Properties of water	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	Density of water (ρ)	985.46	979.77	974.00
2.	Kinematic viscosity of water (ν)	$0.478x10^{-6}$	$0.421x10^{-6}$	$0.364x10^{-6}$
3.	Dynamic viscosity of water (μ)	$471x10^{-6}$	$412x10^{-6}$	$354x10^{-6}$
4.	Thermal conductivity of water (k_m)	0.651	0.660	0.668
5.	Prandtl number (Pr)	3.020	2.620	2.220

Table 5.4

Experimental data – Brass material

S.N.	Particulars	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	Diameter of spherical shape (mm)	45	45	45
2.	Characteristic length (mm)	7.5	7.5	7.5
3.	Material of spherical shape	Brass	Brass	Brass
4.	Bath temperature	62	74	82
5.	Initial temperature of shape	27	24	24
6.	Measured temperature (Step change)	60	70	80
7.	Time (during transient heat conduction)	125	140	155

Analysis : BRASS (Diameter 45 mm)



Case – 1
Material – Brass
Shape - Spherical
Initial shape temperature (T_i) = 27°C
Water bath temperature (T_∞) = 62°*C*
Measured temperature (T_0) = 60°C
at time (t) = 130 seconds

Pr*operties* of water at 60°C
Density (ρ) = 985.46 kg/m^3
Kinematic viscosity (ν) = 0.478x10^{-6} m^2/s
Dynamic viscosity (μ) = 471x10^{-6} m^2/*s*
Thermal conductivity (k) = 0.651 W/m°C
Prandtl number (Pr) = 3.02
β = Volumetric expension of the material

$$\beta = \frac{1}{T_f} = \frac{1}{(T_i + T_\infty)/2}$$

Using correlation for free convection

$$\text{Grashof number (Gr)} = \frac{D^3 \rho^2 g \Delta T \beta}{\mu^2}$$

Ra = Gr.Pr
If, Ra $\leq 10^{11}$; Pr ≥ 0.7

By calculation:
Gr =2.97x10^9
Pr = 3.02
So, Ra = 8.97x10^9

$$\text{Nusselt no. (Nu)} = 2 + \frac{0.589 x Ra^{0.25}}{[1+\{\frac{0.469}{Pr}\}^{0.563}]^{0.44}}$$

and Nusselt no. (Nu) = 135

$$\text{Heat transfer coefficient (h)} = \frac{Nu}{D} k_w = \frac{135}{45x10^{-3}} * 0.651 = 1953 \frac{W}{m^2\ °C} \qquad (1)$$

Dimensionless temperature $(\theta_0) = \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.0345$

Fourier number $(F_o) = \frac{\alpha}{D^2} t = 4.0$, time t = 220 sec.

From Heisler chart

Biot number (Bi) = 0.308

Biot number $(Bi) = \frac{hxL_c}{k_m}$

Thermal conductivity of material $(k_m) = \frac{L_c}{Bi} h$

$(k_m) = \frac{7.5x10^{-3}}{0.308} h = 0.024$ xh..(2)

$(k_m) = 54.00 \frac{W}{m\ ^\circ C}$

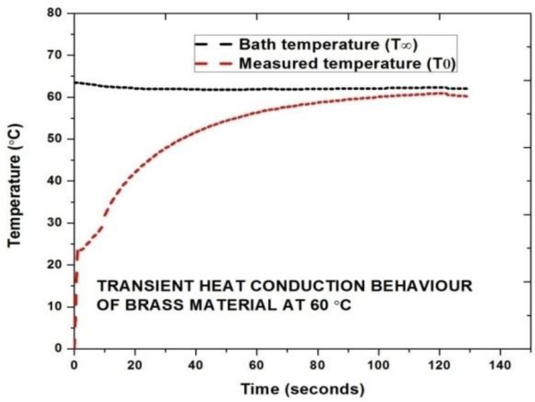


Figure 5.2**:** Transient heat conduction behavior of steel material at 60°C

VI. EXPERIMENAL RESULTS AND DISCUSSION

Table 6.1

Heat transfer due to conduction, convection and radiation of Steel and Brass material

Steel material

S.N.	Particulars	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	At conduction: $Q_{cond.}$ (W)	504	675	855
2.	At convection: $Q_{conv.}$ (W)	410	502	806
3.	At radiation: Q_{rad} (W)	1.3	1.84	2.4
4.	**Q_{total} (W)**	**915.7**	**1178.8**	**1663.4**

Brass material

S.N.	Particulars	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	At conduction: $Q_{cond.}$ (W)	1025	1468	1740
2.	At convection: $Q_{conv.}$ (W)	410	502	806
3.	At radiation: Q_{rad} (W)	1.28	1.87	2.38
4.	Total amount of heat transfer **Q_{total} (W)**	**1436.28**	**1971.87**	**2548.38**

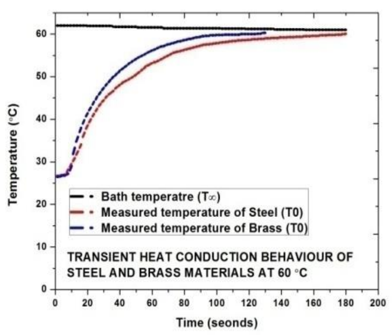
Table 6.2

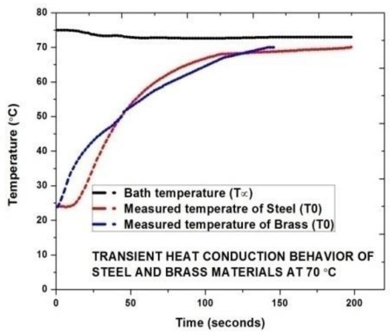
Natural heat transfer coefficient, thermal conductivity (km) of Steel and Brass materials at 60, 70 and 80°C

S.N.	Particular	Case-1 (60°C)	Case-2 (70°C)	Case-3 (80°C)
1.	Heat transfer coefficient (h) $\frac{W}{m^2\ °C}$	1953	1718	2264
2.	Thermal conductivity of Steel (Km) $\frac{W}{m\ °C}$	54	52	54
3.	Thermal conductivity of Brass (Km) $\frac{W}{m\ °C}$	110	113	110

Plots transient heat conduction behavior of Steel and Brass materials at 60°C, 70°C and 80°C

Case-1(at 60°C)	Case-1(at 70°C)	Case-1(at 80°C)





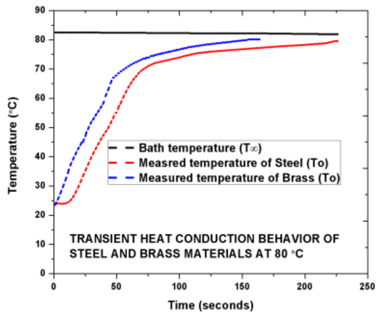
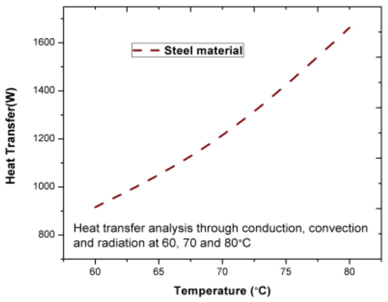
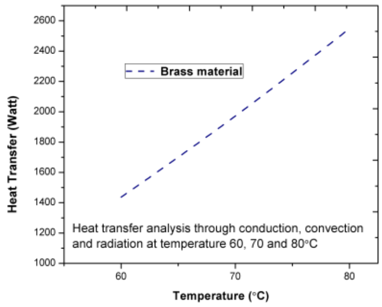


Figure 6.1: Transient heat conduction behavior of Steel and Brass materials at 60°C, 70°C and 80°C

Plots heat transfer v/s temperature for Steel and Brass material

Steel material	Brass material	Steel and Brass material





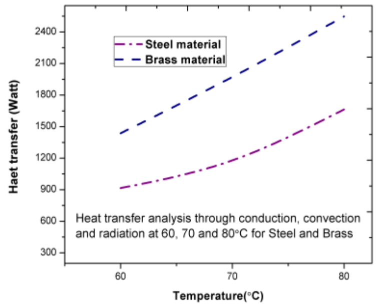


Figure 6.2: Heat transfer due to conduction, convection and radiation in Steel and Brass materials at 60°C, 70°C and 80°C

Plots thermal conductivity v/s temperature for Steel and Brass materials

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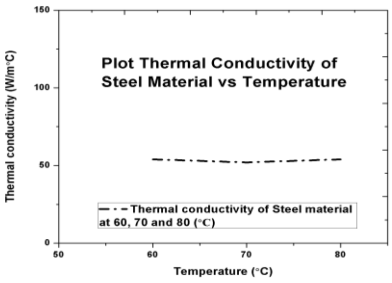
<td>
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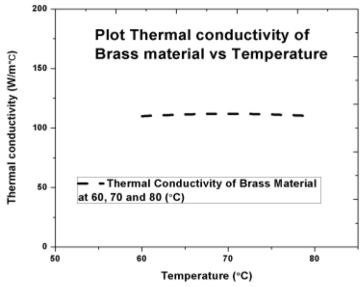
<td>
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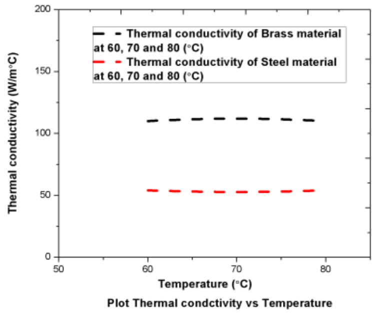
<td>
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Figure 6.3: Thermal conductivity of Steel and Brass materials at 60°C, 70°C and 80°C

VII. RESULTS &CONCLUSION

A spherical shape is used which thermal conductivity is to be measured. A 'K' type of thermocouple is used due to environmental consideration, low price, easily availability and accurate measurement of temperature. The shape is immersed into the constant temperature water bath, maintained at particular temperature. Bead of the thermocouple is located at the centre of the sphere for its temperature measurement during transient heat conduction. Data is recorded by using data acquisition system.

Subsequently, few dimensionless numbers like Prandtl (Pr), Grashof (Gr), Rayleigh (Ra), Nusselt (Nu) have been found by using the properties of water at particular temperature. Based on these dimensionless numbers, natural heat transfer coefficient (h) has been found in the range of 1700 to 2264 $\frac{W}{m^2\ °C}$. Due to transient heat conduction through the shape, some part of heat is also absorbed. So, this has been found that Fourier number (Fo) increases with the bath temperature. It is in the range of 3.30 to 4.00 for steel material and 7.10 to 8.49 for brass material. The values of Biot numbers (Bi) are as 0.27 to 0.35 for steel and 0.1136 to 0.1176 for brass material.

Thermal conductivity of both Steel and Brass materials have almost been found constant during temperature range 55°C to 90°C and their values are 52 to 54 and 110 to 113 $\frac{W}{m\ °C}$ for Steel and Brass respectively.

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