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LOGARITHMIC COEFFICIENTS IMMINENT TO CONVEX FUNCTIONS

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***ABSTRACT*. For f scientific and near curved in D = {z: |z|<1}, we give sharp gauges for the logarithmic coefficients γn of f characterized by log $\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n$ when n = 1, 2, 3.**

***Keywords*: logarithmic coefficients, convex functions**

INTRODUCTION

Give S a chance to be the class of standardized expository univalent f for $z \in$ D = $\{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The logarithmic coefficients of f are characterized in D by

$$\text{Log}\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n$$

The logarithmic coefficients γ_n assume a focal part in the hypothesis of univalent capacities. Milin guessed that for $f \in S$ and n ≥ 2,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma k|^2 - \frac{1}{k}\right) \le 0,$$

what's more, it isn't hard to see that (2) infers the Bieberbach guess. It was a proof of (2) that De Branges built up keeping in mind the end goal to demonstrate the guess.

Not very many correct upper limits for γ_n appear have been built up, With more consideration being given to aftereffects of a normal sense (see e.g [1], [2]). Additionally it is realized that for $f \in S$, the normal disparity $|\gamma_n| \le \frac{1}{n}$ is false even arranged by extent [1]

Differentiating (1) and equating coefficients gives

(1) $$\gamma_1 = \frac{1}{2}a_2,$$

(2) $$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right)$$

(3) $$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\right).$$

Thus |γ_1 | ≤ 1 takes after on the double from (3), and utilization of the Fekete-Szego inquality in (4), [1, Theorem 3.8], gives the sharp gauge

$$|\gamma_2| \le \frac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

For n ≥ 3, the issue appears to be substantially harder, and no critical upper limits for $|\gamma_n|$ when $f \in S$ give off an impression of being known.

Signify by S* the subclass of S of starlike capacities, with the goal that $f \in S^*$ if and if, for $z \in D$,

$$Re\,\frac{zf'(z)}{f(z)} > 0.$$

Thus we can write $zf'(z) = f(z)h(z)$, where $h \in P$, the class of functions satisfying $Re\, h(z) > 0$ for $z \in D$. By simple differentiation in (1) again and noting that the coefficients c_n of the Taylor series of h about z = 0 satisfy $|c_n| \le 2$ for $n \ge 1$ shows that $|\gamma_n| \le \frac{1}{n}$ holds for $f \in S^*$ and $n \ge 2$.

Suppose now that f is analytic in D. Then f is close to convex if and only if, for $z \in D$, there exists $g \in S^*$ such that

(6) $$Re\,\frac{zf'(z)}{g(z)} > 0.$$

We signify the class of near curved capacities by K and note the notable incorporation relationship $S^* \subset K \subset S$.

That the imbalance $|\gamma_n| \le 1/n$ for n ≥ 2 reaches out to the class K was asserted in a paper of Elhosh [3]. Anyway Girela [4] brought up a mistake in the confirmation and demonstrated that for $f \in K$, this disparity is false for n ≥ 2. In a similar paper it was demonstrated that $|\gamma_n| \le 3/2n$ holds for n ≥ 1 at whatever point f has a place with the arrangement of the extraordinary purposes of the shut curved body of the class K, which infers that $|\gamma_3| \le 1/2$ for this situation. As was called attention to over, this headed is false for the whole class K. It is the reason for this paper to set up the sharp bound $|\gamma_3| \le 7/12$ for the class K when the coefficient b2 in the Taylor extension for g(z) is genuine.

We first note that from (4) it is a prompt result of the Fekete-Szego imbalance for $f \in K$ [5] that the accompanying sharp disparity holds for $f \in K$:

$$|\gamma_2| = \frac{1}{2}|a_3 - \frac{1}{2}a_2^2| \leq \frac{11}{18} = 0.6111.$$

We now turn our attention to the case $n = 3$ for the class K.

It follows from (6) that we can write $zf'(z) = g(z)h(z)$, where $Re\ h(z) > 0$ for $z \in D$ and, since $g \in S^*$, $zg'(z) = g(z)p(z)$, where $Re\ p(z) > 0$ for $z \in D$.

Now write

$$(7) \qquad h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

$$(8) \qquad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

$$(9) \qquad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

We shall need the following result [6], which has been used widely.

Lemma. *Let h, p $\in$ P and be given by* (7) *and* (8) *respectively. Then for some complex valued x with* $|x| \leq 1$ *and some complex valued t with* $|t| \leq 1$,

$$2c_2 = c_1^2 + x(4 - c_1^2),$$
$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t.$$

Similarly for some complex valued y with $|y| \leq 1$ *and some complex valued s with* $|s| \leq 1$,

$$2p_2 = p_1^2 + y(4 - p_1^2),$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 y - p_1(4 - p_1^2)y^2 + 2(4 - p_1^2)(1 - |y|^2)s.$$

We prove the following :

Theorem. Let $f \in K$, *then*

$$|\gamma_1| \leq 1, \qquad |\gamma_2| \leq \frac{11}{18}.$$

Also when $f \in K$ and b_2 is real,

$$|\gamma_3| \leq \frac{7}{12}.$$

The inequalities are sharp.

Proof. As noted above, the first two inequalities are proved. Thus it remains to prove the third.

From (5) we need to find an upper bound for

$$(10) \qquad |\gamma_3| = \frac{1}{2}|a_4 - a_2 a_3 + \frac{1}{3}a_2^3|.$$

First note that by equating coefficients we have

$$2a_2 = c_1 + p_1,$$
$$3a_3 = c_2 + c_1 p_1 + \frac{p_1^2 + p_2}{2},$$
$$4a_4 = c_3 + c_2 p_1 + \frac{c_1(p_1^2 + p_2)}{2} + \frac{p_1^3}{6} + \frac{p_1 p_2}{2} + \frac{p_3}{3}.$$

Substituting into (10) gives

$$(11) \qquad |a_4 - a_2 a_3 + \frac{1}{3}a_2^3| = |\frac{c_3}{4} + \frac{c_2 p_1}{12} + \frac{c_1 p_2}{24} + \frac{p_3}{12} + \frac{p_1 p_2}{24} - \frac{c_1 c_2}{6} - \frac{c_1^2 p_1}{24} + \frac{c_1^3}{24}|.$$

We now use the Lemma to eliminate c_2, c_3, p_2 and p_3 from (11) and obtain

$$(12) \qquad |a_4 - a_2 a_3 + \frac{1}{3}a_2^3| = |\frac{c_1^3}{48} + \frac{c_1 x X}{24} - \frac{c_1 x^2 X}{16} + \frac{XZ}{8} + \frac{p_1 x X}{24} + \frac{c_1 p_1^2}{48} + \frac{c_1 y Y}{48} + \frac{p_1 y Y}{16} - \frac{p_1 y^2 Y}{48} + \frac{p_1^3}{24} + \frac{YV}{24}|$$

where, for simplicity, we have set $X = 4 - c^2{}_1$, $Y = 4 - p^2{}_1$, $Z = (1 - |x|^2)s$ and $V = (1 - |y|^2)t$.

Without loss of generality we may write $c_1 = c$ with $0 \le c \le 2$. Also, since we are assuming $b_2 = p_1$ to be real, we can write $p_1 = q$, with $0 \le |q| \le 2$. Writing $|q| = p$, it then follows using the triangle inequality in (12) together with $|s| \le 1$ and $|t| \le 1$ that

$$
\begin{aligned}
|a_4 - a_2 a_3 + \tfrac{1}{3}a_2^3| &\le \frac{c^3}{48} + \frac{c|x|X}{24} + \frac{c|x|^2X}{16} + \frac{XZ}{8} + \frac{p|x|X}{24} + \frac{cp^2}{48} \\
&\quad + \frac{c|y|Y}{48} + \frac{p|y|Y}{16} + \frac{p|y|^2Y}{48} + \frac{p^3}{24} + \frac{YV}{24} \\
&= F(c, p, |x|, |y|),
\end{aligned} \tag{13}
$$

say, where now $X = 4 - c^2$, $Y = 4 - p^2$, $Z = 1 - |x|^2$ and $V = 1 - |y|^2$.

Thus we need to find the maximum of $F(c, p, |x|, |y|)$ over the hyper-rectangle $R = [0, 2] \times [0, 2] \times [0, 1] \times [0, 1]$.

From (13), substituting for X, Y, Z and V gives

$$
\begin{aligned}
F(c, p, |x|, |y|) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}c|x|(4-c^2) + \frac{1}{16}c|x|^2(4-c^2) \\
&\quad + \frac{1}{8}(4-c^2)(1-|x|^2) + \frac{1}{24}p|x|(4-c^2) + \frac{1}{48}c|y|(4-p^2) \\
&\quad + \frac{1}{16}p|y|(4-p^2) + \frac{1}{48}p|y|^2(4-p^2) + \frac{1}{24}(4-p^2)(1-|y|^2).
\end{aligned} \tag{14}
$$

We first assume that $F(c, p, |x|, |y|)$ has a maximum value at an interior point $(c_0, p_0, |x_0|, |y_0|)$ of R. Then since

$$
\frac{\partial F}{\partial |x|} = \frac{1}{24}c(4-c^2) + \frac{1}{4}c|x|(4-c^2) - \frac{1}{4}|x|(4-c^2) + \frac{1}{2}p(4-c^2) = 0
$$

at such a point, it follows that $c_0 = 2$, which is a contradiction. Hence any maximum points must be on the boundary of R.

Thus we need to find the maximum value of $F(c, p, |x|, |y|)$ on each of the 32 edges and 24 faces (8 of co-dimension 1 and 16 of co-dimension 2) of R. Finding these maximum values involves a great many tedious exercises in elementary calculus, and, for the sake of brevity, we omit many of the simple ones. The process does however identify the maximum value of 7/6 needed in the Theorem and shows that the maximum value on all edges and faces is less than or equal to 7/6.

Finding the most extreme estimations of F(c, p, |x|, |y|) on each of the 32 edges includes paltry activities, and demonstrates that F(c, p, |x|, |y|) ≤ 7/6 on these edges. On the 16 countenances of co-measurement 2, comparative basic activities in rudimentary math again demonstrate that F(c, p, |x|, |y|) ≤ 7/6 on every one of these appearances. We along these lines consider the 8 appearances of co-measurement 1 as takes after. On the face $c = 0$, suppose that $|x| \le 1$ in (14), which gives a resulting expression

$$
\begin{aligned}
G_1(0, p, |y|) &= \frac{1}{24}p^3 + \frac{1}{2} + \frac{1}{6}p + \frac{1}{16}p|y|(4-p^2) \\
&\quad + \frac{1}{48}p|y|^2(4-p^2) + \frac{1}{24}(4-p^2)(1-|y|^2).
\end{aligned}
$$

Differentiating $G_1(0, p, |y|)$ with respect to $|y|$ shows that any maximum must occur on the boundary of $[0, 2] \times [0, 1]$, and since the largest value at the end points is 7/6, $F(c, p, |x|, |y|)$ has maximum 7/6 on the face $c = 0$.

On the face $c = 2$, suppose again that $|x| \le 1$ in (14), to obtain the expression

$$
\begin{aligned}
G_2(2, p, |y|) &= \frac{1}{6} + \frac{1}{24}p^2 + \frac{1}{24}p^3 + \frac{1}{24}|y|(4-p^2) \\
&\quad + \frac{1}{16}p|y|(4-p^2) + \frac{1}{48}p|y|^2(4-p^2) + \frac{1}{24}(4-p^2)(1-|y|^2).
\end{aligned}
$$

Following the same procedure gives a maximum of 0.696 on $[0, 2] \times [0, 1]$.

On the face $p = 0$, suppose that $|x| \le 1$ and $|y| \le 1$ in (14), to obtain the expression.

$$
G_3(c, 0, |y|) = \frac{1}{48}c^3 + \frac{5}{48}c(4-c^2) + \frac{1}{8}(4-c^2) + \frac{1}{12}c + \frac{1}{6},
$$

which has maximum value 23/24 on $[0, 2] \times [0, 1]$.

On the face $p = 2$, (14) becomes

$$
\begin{aligned}
G_4(c, 2, |x|) &= \frac{1}{3} + \frac{c}{12} + \frac{1}{48}c^3 + \frac{1}{12}(4-c^2)|x| + \frac{1}{24}c(4-c^2)|x| \\
&\quad + \frac{1}{16}c(4-c^2) + \frac{1}{8}(4-c^2)(1-|x|^2).
\end{aligned}
$$

Differentiating $G_4(c, 2, |x|)$ with respect to $|x|$ and considering the end points gives a maximum value 1.005 on $[0, 2] \times [0, 1]$.

On the face $|x| = 0$, suppose that $|y| \le 1$ in (14), to obtain

$$
\begin{aligned}
G_5(c, p, 0) &= \frac{1}{48}c^3 + \frac{1}{8}(4-c^2) + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}(4-p^2) \\
&\quad + \frac{1}{12}p(4-p^2) + \frac{1}{48}c(4-p^2).
\end{aligned}
$$

It is now an easy exercise to show that $G_5(c, p, 0)$ has a maximum value of 0.9531 when $p = 4/3$ and $c = 2 - 2\sqrt{6}/3$ on $[0, 2] \times [0, 2]$.

On the face $|x| = 1$, suppose that $|y| \le 1$ in (14), to obtain

$$\begin{aligned} G_6(c,p,1) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{5}{48}c(4-c^2) + \frac{1}{24}p(4-c^2) \\ &+ \frac{1}{24}(4-p^2) + \frac{1}{12}p(4-p^2) + \frac{1}{48}cp(4-p^2) \\ &= \frac{1}{6} + \frac{1}{2}c - \frac{1}{12}c^3 + \frac{1}{2}p - \frac{1}{24}c^2p - \frac{1}{24}p^2 - \frac{1}{24}p^3 \\ &\le \frac{1}{6} + \frac{1}{2}c - \frac{1}{12}c^3 + \frac{1}{2}p - \frac{1}{24}p^3. \end{aligned}$$

It is now a simple exercise to show that this expression has maximum value 5/6 on [0, 2] × [0, 2].

On the face $|y| = 0$, (14) becomes

$$\begin{aligned} G_7(c,p,|x|) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}(4-p^2) \\ &+ \frac{1}{24}c|x|(4-c^2) + \frac{1}{24}p|x|(4-c^2) + \frac{1}{16}c|x|^2(4-c^2) \\ &+ \frac{1}{8}(4-c^2)(1-|x|^2). \end{aligned}$$

Differentiating $G_7(c, p, |x|)$ with respect to $|x|$ shows, as before, that there are no maximum points in the interior of [0, 2] × [0, 2] × [0, 1], and so we need only find the maximum values of $G_7(c, p, |x|)$ on the boundary of [0, 2]×[0, 2]×[0, 1]. In the interest of brevity, we omit the simple analysis which gives maximum value of 1.005 when $p = 2$ and $|x| = 1$

We finally note that on the face $|y| = 1$,

$$\begin{aligned} G_8(c,p,|x|) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}c|x|(4-c^2) + \frac{1}{16}c|x|^2(4-c^2) \\ &+ \frac{1}{8}(4-c^2)(1-|x|^2) + \frac{1}{24}p|x|(4-c^2) + \frac{1}{48}c(4-p^2) \\ &+ \frac{1}{12}p(4-p^2). \end{aligned}$$

As before, differentiating $G_8(c, p, |x|)$ with respect to $|x|$ shows that there are no maximum points in the interior of [0, 2]×[0, 2]×[0, 1], and so we need only find the maximum values of $G_8(c, p, |x|)$ on the boundary of [0, 2] × [0, 2] × [0, 1]. Again in the interest of brevity, we omit the simple analysis which gives a maximum value of 1.052, again less than 7/6.

Thus we have shown that in all cases, the maximum value of (14) is at most 7/6, which completes the proof of the Theorem.

We finally note that equality in the inequality in $|\gamma_3| \le 7/12$ is attained when $c_1 = 0$ and $c_2 = c_3 = p_1 = p_2 = p_3 = 2$.

Remark 1. The condition that b_2 is real in the inequality for $|\gamma_3|$ arises in order to maximise (12). We conjecture that this condition can be removed and $|\gamma_3| \le 7/12$ for $f \in K$.

Remark 2. The correct growth rate for γ_n appears to be unknown for close to convex functions, and in this direction the best known estimate to date appears to be that of Ye [7], who showed that $|\gamma_n| \le \frac{A \log n}{n}$, where A is an absolute constant.

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