UNPERTURBED FRONTIERS ON PROTUBERANT MAPPINGS IN NUMEROUS COMPLEX VARIABLES

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***ABSTRACT*: A scholarly exposition of some frontiers of convex mappings in several complex variables, which seeks to give an overview of the mathematical faith in theorems as they understand it as well as accessible summaries of its main features. Relied on some of the greatest teachers and adepts to explain its essence. Our narrative weaves between the mathematical concept concrete ideas and an attentive reading of the relevant scriptures and academic texts. We seek to arrive what the reader will find in a lucid and reflective account of one of the world's oldest and greatest faiths and its contemporary existence.**

***Keywords*: Complex variables; Growth and Distortion Theorems**

INTRODUCTION

The coefficient boondocks and the Growth and Distortion Theorems for arched capacities in a single complex variable are summed up to a few factors. The holomorphic mappings considered are characterized in the unit ball or some other space of one of the initial three established composes. Each mapping show cases its area onto an arched set in a one-to – one form. The organize elements of each mapping have multivariable power arrangement about the birthplace. The most ideal upper boondocks are found for specific blends of the coefficients of these power arrangement. On the off chance that the space is the unit circle in the plane, these wildernesses diminish to the traditional coefficient gauges for curved capacities. As an application, these coefficient boondocks are utilized to get the most ideal upper and lower outskirts on the development of the size of each mapping as far as the extent of the autonomous variable. Likewise, gauges on the extents of different subordinates of each mapping are found.

Beginning with strategies which are standard for the Loewner hypothesis of arched elements of one complex variable [3], we will stretch out that hypothesis to a few factors. A portion of our outcomes have been in combined freely, utilizing diverse strategies, by T. Suffridge [5] and T.S. Liu [2] and J. Pfalzgraff.

1. **Notation.** We will use the following standard notation for several complex variables. A point in C^n will be denoted by a column vector.

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

and a mapping *f(z)* from C^n to C^n will be denoted by

$$f(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \\ \vdots \\ f_n(z) \end{pmatrix}$$

where each coordinate function f_k is a function from C^n to C. The complex Jacobian of f at z, that is,

$$\left(\frac{\partial f_p}{\partial z_q}\right)_{p,q=1}^{n}$$

will be denoted by $J_f(z)$.

We will consider normalized convex mappings from C^n to C^n. A convex mapping is a mapping with range a convex set. Let *f(z)* be a one-to-one convex mapping from

$$B^n = \left\{ z : |z| = \sqrt{|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2} < 1 \right\}$$

into C^n. We wish to normalize *f(z)* so that $f(0) = 0$ and $J_f(0) = I_n$, the n-dimensional identity matrix. Note that this can be done because since f is one-to-one, $J_f(0)$ is invertible. The normalization takes place by a complex affine transformation, $Jf(0)^{-1}[f(z) — f(0)]$. This complex affine transformation preserves the convexity of the range. Then f has the form

$$f(z)=\begin{pmatrix} z_1+\sum\limits_{|p|>1} d_p^{(1)} z^p \\ z_2+\sum\limits_{|p|>1} d_p^{(2)} z^p \\ \vdots \\ z_n+\sum\limits_{|p|>1} d_p^{(n)} z^p \end{pmatrix}$$

where the sums are over vector indices $p = \{p_1, p_2, \ldots\ldots, p_n)$ with $|p| = p_1+\ldots+p_n$ and $z^p = z_1^{p1} z_2^{p2} \ldots z_n^{pn}$, with each p_k a nonnegative integer.

2. Some best possible frontiers for convex mappings.

LEMMA 2.1. *Let f(z) be a normalized mapping from* B^n *to* C^n *of the above form. Let* $1 \le r \le n$, $m > 1$, *and* $m \in Z$. *Then*

$$\sum_{r=1}^{n} \left| d_{(m,0,\ldots,0)}^{(r)} \right|^2 \le 1 \ .$$

Proof. Let $= e^{\frac{2\pi t}{m}}$. since $\sum_{t=0}^{m-1} \varepsilon^{kt} = m$ if m divides k and =0 otherwise,

$$\sum_{t=0}^{m-1} f\begin{pmatrix} z_1^{\frac{1}{m}}\varepsilon^t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left(m \sum_{s=1}^{\infty} d_{(ms,0,\ldots,0)}^{(r)} z_1^s \right)_r$$

where r runs from 1 to n, and indicates the components of the vector. Let

$$h(z_1)=\begin{pmatrix} h_1(z_1) \\ h_2(z_1) \\ \vdots \\ h_n(z_1) \end{pmatrix} = f^{-1}\left(\frac{1}{m} \sum_{t=0}^{m-1} f\begin{pmatrix} z_1^{\frac{1}{m}}\varepsilon^t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

The right side is defined for it is the inverse image of a convex conglomeration of points in the convex range of f. The initial term of the r-th component of $h(z_1)$ can be found by observing that since f behaves near the origin like the identity mapping, so does f^{-1}. Thus, $h(z_1) = d_{(m,0,\ldots 0)}^{(r)} z_1 + \cdots)_r \in B^n$.

Hence the component function $h_r(z_1)$ is an analytic function from the unit disk to itself with $h_r(0) = 0$. Since $h(z_1) \in B^n$, $|h(z_1)|^2 < 1$.

Let

$$\begin{aligned} q(z_1) &= e^{i\varphi_1} h_1(z_1)^2 + \ldots + e^{i\varphi_n} h_n(z_1)^2 \\ &= \left[\sum_{r=1}^{n} e^{i\varphi_r} \left(d_{(m,0,\ldots,0)}^{(r)} \right)^2 \right] z_1^2 + \ldots \ . \end{aligned}$$

Consider $\frac{q(z_1)}{z_1^2}$. The initial terms of the series put under the scanner expansion for $h_r\{z_1)$, we visualize that the singularity of $\frac{q(z_1)}{z_1^2}$ at the origin is removable.

For any given $0 < \varepsilon < 1$, consider $|z_1| = 1 — \frac{1}{4}\varepsilon$. Then $\left|\frac{q(z_1)}{z_1^2}\right| \le 1 + \varepsilon$. Radiated by the maximum principle, this inequality holds for $|z_1|$ - 1 — $\frac{1}{4}\varepsilon$. In particular, at $z_1 = 0$,

$$\left| \sum_{r=1}^{n} e^{i\varphi_r} \left(d_{(m,0,\ldots,0)}^{(r)} \right)^2 \right| \le 1 + \varepsilon \ .$$

Choose $\varphi_1, \ldots, \varphi_n$ so that

$$\left| \sum_{r=1}^{n} e^{i\varphi_r} \left(d_{(m,0,\ldots,0)}^{(r)} \right)^2 \right| = \sum_{r=1}^{n} \left| d_{(m,0,\ldots,0)}^{(r)} \right|^2$$

Since the former expansion is $\le 1 + \varepsilon$ for all $0 < \varepsilon < 1$, the last conglomeration of coefficients is ≤ 1, as claimed.

We now estimate the growth of f .

PROPOSITION 2.1. *Let f(z) be a normalized convex mapping from* B^n *into* C^n. *Let U be a unit vector, and let* $0 \le r < 1$. *Then*

$$|f(rU)| \le \frac{r}{1-r} \ .$$

Proof. Rotate the domain so that

$$rU = \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By the triangle inequality

$$\left| f\begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|^2 = \left| \sum_{r=1}^{n} \sum_{k,p=1}^{\infty} d^{(r)}_{(k,0,\dots,0)} \overline{d^{(r)}_{(p,0,\dots,0)}} z_1^k \overline{z_1^p} \right|$$
$$\leq \sum_{r=1}^{n} \sum_{k,p=1}^{\infty} \left| d^{(r)}_{(k,0,\dots,0)} \right| \left| d^{(r)}_{(p,0,\dots,0)} \right| |z_1|^{k+p}$$
$$= \sum_{m=2}^{\infty} \left(\sum_{k+p=m,\, k,p\geq 1} \left(\sum_{r=1}^{n} \left| d^{(r)}_{(k,0,\dots,0)} \right| \left| d^{(r)}_{p,0,\dots,0)} \right| \right) \right) |z_1|^m .$$

Radiated by Cauchy's inequality;

$$\sum_{r=1}^{n} \left| d^{(r)}_{(k,0,\dots,0)} \right| \left| d^{(r)}_{(p,0,\dots,0)} \right| \leq \sqrt{\sum_{r=1}^{n} \left| d^{(r)}_{(k,0,\dots,0)} \right|^2} \sqrt{\sum_{s=1}^{n} \left| d^{(s)}_{(p,0,\dots,0)} \right|^2} \leq 1 ,$$

By Lemma 2.1. Hence

$$\left| f\begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|^2 \leq \sum_{m=2}^{\infty} \sum_{k+p=m,\, k,p\geq 1} |z_1|^m$$
$$= \sum_{k=2}^{\infty} (k-1)|z_1|^k = \frac{|z_1|^2}{(1-|z_1|)^2} .$$

Taking the square root, we obtain

$$\left| f\begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \leq \frac{|z_1|}{1-|z_1|} ,$$

And the conclusion of the proposition.
Note that this upper bound is attained by the following mapping :

$$f(z) = \begin{pmatrix} \dfrac{z_1}{1-z_1} \\ \dfrac{z_2}{1-z_1} \\ \vdots \\ \dfrac{z_n}{1-z_1} \end{pmatrix}$$

This mapping can be understood as follows: Consider the Cayley transform of the ball onto the generalized half plane. Clearly that transform is convex. In the pursuit normalization, it is *f(z)* and is still a convex mapping. [1]

PROPOSITION 2.2. *Let f(z) be a normalized convex mapping from B^n into C^n . Let U be a unit vector, let* $0 \leq r \leq 1$, *and let t be a positive integer. Then*

$$\left| D_U^t f(rU) \right| \leq \frac{t!}{(1-r)^{t+1}} .$$

Proof. It can be assumed that

$$rU = \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then

$$\left|\frac{\partial^t}{\partial z_1^t} f \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right|^2 = \sum_{r=1}^{n} \left|\sum_{k=t}^{\infty} d^{(r)}_{(k,0,\dots,0)} z_1^{k-t} k(k-1)\dots(k-t+1)\right|^2$$

$$= \sum_{r=1}^{n} \left(\sum_{k=t}^{\infty} k(k-1)\dots(k-t+1) d^{(r)}_{(k,0,\dots,0)} z_1^{k-t}\right) \cdot \left(\sum_{p=t}^{\infty} p(p-1)\dots(p-t+1)\overline{d^{(r)}_{(p,0,\dots,0)} z_1^{p-t}}\right)$$

$$= \sum_{r=1}^{n} \sum_{k,p=t}^{\infty} k\dots(k-t+1)p\dots(p-t+1) d^{(r)}_{(k,0,\dots,0)} \overline{d^{(r)}_{(p,0,\dots,0)}} z_1^{k-t} \overline{z_1^{p-t}}$$

By the triangle inequality,

$$\leq \sum_{r=1}^{n} \sum_{k,p=1}^{\infty} k\cdots(k-t+1)p\cdots(p-t+1) \cdot \left|d^{(r)}_{(k,0,\dots,0)}\right| \left|d^{(r)}_{(p,0,\dots,0)}\right| |z_1|^{k+p-2t}$$

$$= \sum_{m=2t}^{\infty} \sum_{k+p=m,\, k,p\geq t} k\cdots(k-t+1)p\cdots(p-t+1) |z_1|^{m-2t} \sum_{r=1}^{n} \left|d^{(r)}_{(k,0,\dots,0)}\right| \left|d^{(r)}_{(p,0,\dots,0)}\right| .$$

By using Cauchy's inequality, one can see that

$$\leq \sum_{m=2t}^{\infty} \sum_{k+p=m,\, k,p\geq t} k\dots(k-t+1)p\dots(p-t+1)|z_1|^{m-2t} \sqrt{\sum_{r=1}^{n} \left|d^{(r)}_{(k,0,\dots 0)}\right|^2} \sqrt{\sum_{r=1}^{n} \left|d^{(r)}_{(p,0,\dots,0)}\right|^2} .$$

By Lemma 2.1, each of the radicals is bounded by one.

$$\leq \sum_{m=2t}^{\infty} \sum_{k+p=m,\, k,p\geq t} k\dots(k-t+1)p\dots(p-t+1)|z_1|^{m-2t} = \left(\frac{t!}{(1-|z_1|)^{t+1}}\right)^2 .$$

Taking the square root of both sides of the inequality, we obtain the desired estimate. Again these estimates are best possible since the normalized Cayley transform attains the upper bound at each point of the polar ray.
For the next result, we need the following lemma.
LEMMA 2.2. If *f(x)* is continuous on [*a*,*b*] and

$$\liminf_{\Delta\to 0^+} \frac{f(x+\Delta)-f(x)}{\Delta} \geq 0$$

for each $a \leq x < b$, then $f(b) \geq f(a)$.

Proof. Consider $g(x) = f(x) + \varepsilon x$ for x in [α, *b*] and ε a positive constant. Then

$$\liminf_{\Delta\to 0^+} \frac{g(x+\Delta)-g(x)}{\Delta} = \liminf_{\Delta\to 0^+} \frac{f(x+\Delta)+\varepsilon(x+\Delta)-f(x)-\varepsilon x}{\Delta} = \liminf_{\Delta\to 0^+} \frac{f(x+\Delta)-f(x)}{\Delta} + \varepsilon > 0 .$$

Let c be a point where $g(x)$ attains its maximum value. Suppose that $c < b$. Since

$$\liminf_{\Delta \to 0^+} \frac{g(c+\Delta) - g(c)}{\Delta} > 0 \ ,$$

for $\Delta > 0$ and sufficiently small, $g(c + \Delta) > g(c)$, which contradicts the maximality of $g(c)$. This implies that g has maximum value at b, and $g(b) \geq g(a)$. Thus $f(b) + \varepsilon b \geq f(a) + \varepsilon a$. Take the limit of both sides as ε approaches 0 to obtain $f(b) \geq f(a)$.

PROPOSITION 2.3. *Let f(z) be a normalized convex mapping from B^n into C^n. Let U be a unit vector, and let* $0 \leq r \leq 1$. *Then*

$$|f(rU)| \geq \frac{r}{1+r} \ .$$

Proof. Note that f can be multiplied by a constant complex unitary matrix without changing the conclusion. Assume that f is a convex mapping from B^n to C^n, with $f(0) = 0$, and with $J_f(0)$ unitary.
By rotating the domain, we can let

$$\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with $a > 0$, be a point in $\{z: |z| = 1\}$ at which $|f(z)|$ is minimized. Since $Jf(0)$ was only assumed to be unitary, we can also rotate the range so that

$$f\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By the minimality of

$$\left| f\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|, \quad \frac{\partial f_1}{\partial z_k} = 0$$

For $k = 2,3,\ldots, n$, Thus

$$J_f\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & 0 \ldots 0 \\ * & \ldots * \end{pmatrix}$$

And

$$J_f^{-1}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\frac{\partial f_1}{\partial z_1}} & 0 \ldots 0 \\ * & \ldots * \end{pmatrix}$$

Let $\varphi a(z)$ be a holomorphic automorphism of B^n that maps

$$0 \ \text{to} \ \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ \text{to} \ 0 \ .$$

Then

$$\varphi_a(z) = \begin{pmatrix} \dfrac{z_1 - a}{1 - az_1} \\ \dfrac{z_2\sqrt{1-a^2}}{1 - az_1} \\ \vdots \\ \dfrac{z_n\sqrt{1-a^2}}{1 - az_1} \end{pmatrix} \text{ and}$$

$$J_{\varphi_a}^{-1}(0) = \begin{pmatrix} \dfrac{1}{1-a^2} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{\sqrt{1-a^2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{\sqrt{1-a^2}} \end{pmatrix}$$

Now normalize the mapping. Let

$$F(z) = J_{\varphi_a}^{-1}(0) J_f^{-1}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}\left[f \circ \varphi_a(\zeta) - f\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \dfrac{1}{1-a^2} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{\sqrt{1-a^2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{\sqrt{1-a^2}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\frac{\partial f_1}{\partial z_1}} & 0 \cdots 0 \\ * & \cdots * \end{pmatrix}$$

$$\left(f \circ \varphi_a(\zeta) - \begin{pmatrix} f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$$

Then

$$F\begin{pmatrix} -a \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{-f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{(1-a^2)\dfrac{\partial f_1}{\partial z_1}} \\ * \\ \vdots \\ * \end{pmatrix}$$

The mapping $F(z) = 0$, is a normalized convex mapping, because $F(0)=0$, $J_F(0)= I$, and this normalization process preserves the convexity of the range. Thus by Proposition 2.1,

$$\left| \frac{-f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{(1-a^2)\dfrac{\partial f_1}{\partial z_1}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right| \le \frac{a}{1-a}$$

Since

$$f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq 0 \,,$$

$$\left| \frac{\dfrac{\partial f_1}{\partial z_1}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right| \geq \frac{1}{a(1+a)}$$

And

$$\left| \frac{\partial}{\partial t} \log f_1 \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \geq \frac{1}{a(1+a)} \,,$$

where the last partial derivative is with respect to the real variable t and the expression is then evaluated at $t = a$. Then

$$\frac{\partial f_1}{\partial z_1}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq 0 \quad \text{and} \quad f_1\begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is locally conformal near $z_1 = a$. By the minimizing choice of the point

$$\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad \frac{\dfrac{\partial f_1}{\partial t}\begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix}}{f_1\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$$

Is real and positive. Hence

$$\left| \frac{\partial}{\partial t} \log \overline{f_1 \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right| = \left| \frac{\partial}{\partial t} \log f_1 \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|$$

At $t = a$, and, at $t = a$,

$$\left| \frac{\partial}{\partial t} \log \left| f_1 \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| \right| = \left| \frac{1}{2} \frac{\partial}{\partial t} \log \left| f_1 \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|^2 \right| \geq \frac{1}{a(1+a)} \,.$$

Let $q(r) = \min_{|z|=r} \log|zf(z)|$. For any z_0 with $|z_0|=r$, where the minimum is attained, and for any $\eta > 0$,

$$\min_{\substack{|z|=r+\Delta r \\ \Re\left\langle \frac{z}{|z|}, \frac{z_0}{|z_0|} \right\rangle > 1-\delta}} \log|f(z)| \geq q(r) + (1-\eta)\Delta r \left(\frac{1}{r(1+r)} \right)$$

For δ , $\Delta r > 0$ and both sufficiently small.

Note that the subset of $\{z: |z| = r\}$ on which the minimum of $\log|f(z)|$ is reached is compact. Thus that set of points can be covered by finitely many open spherical caps of $\{z: |z| = r\}$ so that the inequality holds on the related spherical caps on the sphere $\{z: |z| = r + \Delta r\}$. Let Δr_0 be the minimum of the finite number of Δr used. Outside the union of these spherical caps, $\log|f(z)| \geq q(r) + a$ for some fixed α, where $q(r) + a$ is the minimum of $\log|f(z)|$ on the compact set which is the complement of the union of the above open sets. By continuity, a similar inequality holds on the corresponding subset of $|z| = r + \Delta r$. Therefore,

$$q(r + \Delta r) \geq q(r) + (1-\eta)\Delta r \left(\frac{1}{r(1+r)} \right)$$

for $\Delta r_0 \geq \Delta r > 0$. Thus

$$\lim_{\Delta r \to 0^+} \inf \frac{q(r+\Delta r) - q(r)}{\Delta r} \geq \frac{1}{r(1+r)}$$

Then

$$Q(r) \equiv q(r) - \int_{\varepsilon}^{r} \frac{dx}{x(1+x)}$$

satisfies the hypotheses of Lemma 2.2 for $\varepsilon > 0$, and it follows that $Q(r) \geq Q(\varepsilon)$ and hence

$$q(r) - q(\varepsilon) \geq \int_{\varepsilon}^{r} \frac{dr}{r(1+r)} = \int_{\varepsilon}^{r} \left[\frac{1}{r} + \frac{-1}{1+r}\right] dr$$

And

$$q(r) - q(\varepsilon) \geq \log\left[\frac{r}{1+r}\frac{1+\varepsilon}{\varepsilon}\right] .$$

Since $q(r) = \min_{|z|=r} \log|f(z)|$, $q(\varepsilon) = \varepsilon + \ldots$, and

$$\log \frac{|f(z)|}{\epsilon + \ldots} \geq \log\left(\frac{r}{1+r}\right)\left(\frac{1+\varepsilon}{\varepsilon}\right)$$

For $|z| = r$. Thus

$$\log \frac{|f(z)|}{\varepsilon + \ldots} + \log\varepsilon \geq \log\left(\frac{r}{1+r}\right)\left(\frac{1+\varepsilon}{\varepsilon}\right) + \log\varepsilon$$

Allow ε to approach 0, and exponentiate both sides to obtain $|f(z)| \geq (r/(1+r))$.

COROLLARY 2.1. *Let f (z)* be α *convex function from* B^n. *Then f covers the ball of radius* 1/2.

Again, the normalized Cayley transform demonstrates that Proposition 2.3 and Corollary 2.1 give the best possible results.

3. Other frontiers **for Convex Mappings.** The same method can be used to estimate other useful combinations of coefficients. For completeness, we will give some of these results even though they are not necessarily the best possible frontiers.

LEMMA 3.1. *Let f(z) be a normalized convex mapping from* B^n *into* C^n. *Then*

$$\sum_{r=1}^{n} \left| d^{(r)}_{(N,1,0,\ldots,0)} \right|^2 < e(N+1)$$

For N a nonnegative integer.

Proof. Frst consider the case when N = 0. Then

$$\sum_{r=1}^{n} \left| d^{(r)}_{(N,1,0,\ldots,0)} \right|^2 = \sum_{r=1}^{n} \left| d^{(r)}_{(0,1,0,\ldots,0)} \right|^2 = 1 .$$

Now consider $N \geq 1$. Since f has convex range, its range includes the points

$$P(z_1, z_2) = \frac{1}{4N} \sum_{k=1}^{4N} f \begin{pmatrix} e^{\frac{2\pi i k}{4N}} z_1 \\ e^{\frac{-2\pi i k}{4}} z_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

We will consider the contribution of terms of the multiple power series expansion of the coordinate functions of f to the coordinates of the sum $P(z_1, z_2)$.

For terms of f which consist only of a constant, c_1, times a power, p, of the first coordinate variable, the contribution to $P(z_1, z_2)$ is

$$\frac{1}{4N} \sum_{k=1}^{4N} c_1 \left(e^{\frac{2\pi i k}{4N}} z_1\right)^p .$$

Recall that

$$\frac{1}{4N} \sum_{k=1}^{4N} e^{\frac{2\pi i k p}{4N}}$$

is 1 if p is an integer multiple of $4N$ and 0 otherwise. Thus the only terms of this form which contribute to $P(z_1,z_2)$ will be those for which p is an integer multiple of $4N$. Similarly, for terms of f which consist only of a constant times a power, q, of the second coordinate variable, the contribution to $P(z_1,z_2)$ is 0 unless q is an integer multiple of 4.

For mixed terms of f which consist of a constant, c_2, times the first coordinate to a power p and the second coordinate to a power q, the contribution to $P(z_1,z_2)$ is

$$\frac{1}{4N}\sum_{k=1}^{4N} c_2\left(e^{\frac{2\pi ik}{4N}}z_1\right)^p\left(e^{\frac{-2\pi ik}{4}}z_2\right)^q = \frac{1}{4N}\sum_{k=1}^{4N} c_2 e^{\frac{2\pi ik(p-Nq)}{4N}} z_1^p z_2^q .$$

This sum if 0 unless $p - Nq$ is an integer multiple of $4N$. We will consider $z_2 = 0\,(|z_1|^N)$ and will look at terms of the series expansions which are at least $0\,(|z_1|^{2N})$ as $z_1 \to 0$. We will not consider mixed terms with $q > 1$ or $p > 2N - 1$ because such terms are $0\,(|z_1|^{2N})$ Therefore the only mixed terms that will contribute to $P(z_1,z_2)$ are those with $q = 1$ and $p = N$. Thus

$$P(z_1,z_2) = \begin{pmatrix} d^{(1)}_{(N,1,0,\dots,0)} z_1^N z_2 + o\left(|z_1|^{2N}\right) \\ \vdots \\ d^{(n)}_{(N,1,0,\dots,0)} z_1^N z_2 + o\left(|z_1|^{2N}\right) \end{pmatrix}$$

Then $f^{-1}(P(z_1,z_2))$ is defined and in B^n and equals

$$\begin{pmatrix} d^{(1)}_{(N,1,0,\dots,0)} z_1^N z_2 + o\left(|z_1|^{2N}\right) \\ \vdots \\ d^{(n)}_{(N,1,0,\dots,0)} z_1^N z_2 + o\left(|z_1|^{2N}\right) \end{pmatrix}$$

The sum of rotations of the squares of these coordinate functions will have magnitude less than one.

Let

$$\begin{aligned} q(z_1,z_2) &= e^{i\varphi_1}\left[d^{(1)}_{(N,1,0,\dots,0)} z_1^N z_2 + \dots\right]^2 + \dots \\ &\quad + e^{i\varphi_n}\left[d^{(n)}_{(N,1,0,\dots,0)} z_1^N z_2 + \dots\right]^2 \\ &= \left[\sum_{r=1}^{n} e^{i\varphi_r}\left(d^{(r)}_{(N,1,0,\dots,0)}\right)^2\right] z_1^{2N} z_2^2 + o\left(|z_1|^{4N}\right) \end{aligned}$$

Note that q maps B^2 into the unit disk. For a fixed $c > 0$, let $z_2 = cz_1^N$. Then consider

$$Q(z_1) = \frac{q(z_1, cz_1^N)}{z_1^{2N}\left(cz_1^N\right)^2} = \left[\sum_{r=1}^{n} e^{i\varphi_r}\left(d^{(r)}_{(N,1,0,\dots,0)}\right)^2\right] + o(1)$$

Notice that Q(z) has a removable singularity at $z_1 = 0$. Consider that singularity removed. By the maximum principle in one variable,

$$|Q(0)| \le \frac{1}{\sup_{(z_1,cz_1^N)\in\mathbb{B}^2} |c|^2|z_1|^{4N}}$$

That is,

$$\left|\sum_{r=1}^{n} e^{i\varphi_r}\left(d^{(r)}_{(N,1,0,\dots,0)}\right)^2\right| \le \frac{1}{\sup_{(z_1,cz_1^N)\in\mathbb{B}^2} |c|^2|z_1|^{4N}}$$

Since c does not appear on the left side of the preceding inequality, we are free to choose c. Let

$$c = \sqrt{\frac{(N+1)^{N-1}}{N^N}}$$

Then the monotonicity of the right side implies that, if we formally consider the supremum over points (z_1, cz_1^N)in the closure of B^2, the supremum is obtained on the boundary of the closure of B^2, $|z_1|^2 + c^2|z_1|^{2N} \le 1$. The left side of the preceding inequality is monotonic in$|z_1|$, thus there is equality at only one value of $|z_1|$. One value which makes it an equality, and therefore the only solution, is

$$|z_1| = \sqrt{\frac{N}{N+1}}$$

Then

$$|Q(0)| \le (N+1)\left(\frac{N+1}{N}\right)^N$$

Thus

$$\left|\sum_{r=1}^{n} e^{i\varphi_r}\left(d^{(r)}_{(N,1,0,\ldots,0)}\right)^2\right| \le \frac{(N+1)^{N+1}}{N^N}$$

Choose $\varphi_1,\ldots,\varphi_n$ so that

$$\left|\sum_{r=1}^{n} e^{i\varphi_r}\left(d^{(r)}_{(N,1,0,\ldots,0)}\right)^2\right| = \sum_{r=1}^{n}\left|d^{(r)}_{(N,1,0,\ldots,0)}\right|^2$$

Then

$$\sum_{r=1}^{n}\left|d^{(r)}_{(N,1,0,\ldots,0)}\right|^2 \le \frac{(N+1)^{N+1}}{N^N} = \left[1+\frac{1}{N}\right]^N (N+1)$$
$$= \left[1 + N\frac{1}{N} + \frac{N(N-1)}{2}\cdot\frac{1}{N^2} + \ldots + \frac{1}{N^N}\right](N+1)$$
$$< e(N+1)\ .$$

Note: The above value for c is the best choice for c in the preceding proof. This can be demonstrated as follows: Fix any $c > 0$. Then the monotonicity in $|z_1|$ of $c^2|z_1|^{4N}$ implies that

$$\sup_{(z_1, cz_1^N)\in\mathbb{B}^2} c^2|z_1|^{4N}$$

is attained on the boundary of B^2, that is, where $|z_1|^2 + c^2|z_1|^{2N} = 1$. At such a value of $|z_1|$,

(1) $$c^2|z_1|^{4N} = |z_1|^{2N}(1-|z_1|^2)$$

The right side of equation (1) has a fixed value since $|z_1|$ is determined by the point being on the boundary of B^2. Then allow $|z_1|$ to vary in [0,1], and the maximum value of the right side of equation (1) will be greater than or equal to the actual value found using the fixed c. The (maximum ceiling) mentioned above will be attained when the derivative of the right side of equation (1) equals 0. (The endpoints give a value of 0, and are therefore ruled out.) The derivative is 0 when

$$|z_1| = \sqrt{\frac{N}{N+1}}\ ,$$

and then we obtain the same value for the right side of equation (1) as when we chose

$$c = \sqrt{\frac{(N+1)^{N-1}}{N^N}}$$

in the proof of the lemma. Thus the proof of the lemma yields the best possible result for this method of proof. Now Lemma 4.3 will be used to study directional derivatives.

Let $D_V f(z)$ be the directional derivative of $f(z)$ in the direction of the unit vector V. For U and V unit vectors, define U orthogonal to V by orthogonality as complex vectors, that is, $\langle U, V\rangle = 0$.

PROPOSITION 3.1. *Let f(z) be a normalized convex mapping from* B^n *into* C^n *Let U and V be unit vectors with U orthogonal to V as complex vectors, and let* $0 < r < 1$. *Then*

$$|D_V f(rU)| \le \frac{e^{1/2}}{(1-r)^{3/2}}$$

Proof. We can assume that

$$U = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$$

Then

$$\left|\frac{\partial}{\partial z_2} f\begin{pmatrix}z_1\\0\\\vdots\\0\end{pmatrix}\right|^2 = \sum_{r=1}^{n}\left|\sum_{k=0}^{\infty} d^{(r)}_{(k,1,0,\ldots,0)} z_1^k\right|^2$$
$$= \left|\sum_{r=1}^{n}\sum_{k,p=0}^{\infty} d^{(r)}_{(k,1,0,\ldots,0)}\overline{d^{(r)}_{(p,1,0,\ldots,0)}} z_1^k \overline{z_1^p}\right|$$
$$\le \sum_{r=1}^{n}\sum_{k,p=0}^{\infty}\left|d^{(r)}_{(k,1,0,\ldots,0)}\right|\left|d^{(r)}_{(p,1,0,\ldots,0)}\right| |z_1|^k |z_1|^p$$

By the triangle inequality,

$$= \sum_{m=0}^{\infty} \sum_{k+p=m,\, k,p\geq 0} \sum_{r=1}^{n} \left|d^{(r)}_{(k,1,0,\dots,0)}\right| \left|d^{(r)}_{(p,1,0,\dots,0)}\right| |z_1|^m$$

$$\leq \sum_{m=0}^{\infty} \sum_{k+p=m,\, k,p\geq 0} |z_1|^m \sqrt{\sum_{r=1}^{n} \left|d^{(r)}_{(k,1,0,\dots,0)}\right|^2} \sqrt{\sum_{r=1}^{n} \left|d^{(r)}_{(p,1,0,\dots,0)}\right|^2}$$

By Cauchy's inequality

$$\leq \sum_{m=0}^{\infty} \sum_{k+p=m,\, k,p\geq 0} e\sqrt{k+1}\sqrt{m-k+1}|z_1|^m$$

By Lemma 3.1

$$\leq \sum_{m=0}^{\infty} e\sqrt{\sum_{k=0}^{m}(k+1)}\sqrt{\sum_{k=0}^{m}(m-k+1)}|z_1|^m$$

$$= \sum_{m=0}^{\infty} e\left(\frac{(m+1)(m+2)}{2}\right)|z_1|^m = \frac{e}{(1-|z_1|)^3}$$

Taking square roots of both sides, we obtain the desired results.

PROPOSITION 3.2. *Let f(z) be a normalized convex mapping from* B^n *into* C^n. *Let U be a unit vector, and let* $0 < r < 1$. *Then*

$$|\det J_f(rU)| \leq \frac{e^{n-1}}{(1-r)^{(3n+1)/2}}$$

Proof. Assume

$$rU = \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then

$$\left|\frac{\partial}{\partial z_1} f(rU)\right| \leq \frac{1}{(1-|z_1|)^2},$$

And

$$\left|\frac{\partial}{\partial z_k} f(rU)\right| \leq \frac{e^{\frac{1}{2}}}{(1-|z_1|)^{3.2}}$$

for $k = 2,\dots, n$. These are frontiers on the lengths of the columns of J_f, thus

$$|\det J_f(rU)| \leq \frac{1}{(1-|z_1|)^2}\frac{e^{(n-1)/2}}{(1-|z_1|)^{3(n-1)/2}} = \frac{e^{(n-1)/2}}{(1-|z_1|)^{(3n+1)/2}}.$$

4. Convex Matrix Mappings. Now we will consider mappings from the classical domains. Let *f(z) be* a one-to-one biholomorphic mapping from R_I into C^{mxn}, where $m < n$ and

$$R_I = \{z \in \mathbb{C}^{m\times n} : I^{(m)} - \bar{z}z^T > 0\} \qquad [4]$$

where $M > 0$ means that the matrix M is positive semidefinite. For every matrix $M \in C^{mxn}$, define the matrix norm of M by

$$\|M\| = \max_{U\in\mathbb{C}^n, |U|=1} |MU|$$

Lemma 4.1. Let *f(z) be a normalized convex mapping from* R_I *into* C^{mxn}. *Fix*

$$z_0 = \begin{pmatrix} z^0_{11} & 0 & \dots & 0 & \dots & 0 \\ 0 & z^0_{22} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^0_{mm} & \dots & 0 \end{pmatrix}$$

with matrix norm $\|z_0\| = 1$. *Then for all* $\zeta \in D$*, the unit disk in* C, $\zeta z_0 \in R_I$.

Let $g(\zeta) - f(\zeta z_0) = \sum_{n=1}^{\infty} D_n^{jk}\zeta^n$ *,with defined as the coefficient* ζ^n *in the jk-th entry of g. Then for fixed q and j ,*

$$\sum_{k=1}^{n}\left|D_q^{jk}\right|^2 \leq 1 .$$

Proof. Let $\varepsilon = e^{\frac{2\pi 1}{q}}$,and let

$$h(\zeta) = f^{-1}\left(\frac{1}{q}\sum_{t=0}^{q-1} g(\varepsilon^t \zeta^{\frac{1}{q}})\right) = f^{-1}\left(\sum_{p=1}^{\infty} D_{pq}^{jk}\zeta^p\right) = \left(D_q^{jk}\zeta + \dots\right)_{m\times n}$$

(Note that *pq* is a product, unlike *jk*.) This is well-defined because the range is convex. Then for a fixed *j* , since $h(\zeta) \in R_I$,

$$\sum_{k=1}^{n} |h_{jk}(\zeta)|^2 \le 1 \ .$$

Let $k_j(\zeta) = h_{j1}^2(\zeta)e^{i\varphi_{j1}} + h_{j2}^2(\zeta)e^{i\varphi_{j2}} + \dots + h_{jn}^2(\zeta)e^{i\varphi_{jn}}$.

Given any $\varepsilon > 0$, for $|\zeta| < 1 - \frac{1}{4}\varepsilon$,

$$\left|\frac{k_j(\zeta)}{\zeta^2}\right| \le 1 + \varepsilon \ , \qquad \text{thus}$$

$$\left|\sum_{k=1}^{n} e^{i\varphi_{jk}}\left(D_q^{jk}\right)^2\right| \le 1 + \varepsilon \ .$$

Choose $\varphi_{j1}, \dots, \varphi_{jn}$ so that

$$\left|\sum_{k=1}^{n} e^{i\varphi_{jk}}\left(D_q^{jk}\right)^2\right| = \sum_{k=1}^{n}\left|D_q^{jk}\right|^2 \ .$$

Since $\sum_{k=1}^{n}\left|D_q^{jk}\right|^2 \le 1 + \varepsilon$ for any $\varepsilon > 0$, $\sum_{k=1}^{n}\left|D_q^{jk}\right|^2 \le 1$.

PROPOSITION 4.1. *Let f (z) be a normalized convex mapping from* R_I *into* $C^{m\times n}$. *Then*

$$|f(z)| \le \sqrt{m}\frac{\|z\|}{1-\|z\|} \ .$$

Proof. Consider a nonzero *z* in R_I. There exist unitary matrices *U* and *V* such that *UzV* is of the form of z_0 above. There is a positive b such that *bUzV* has norm 1. Then $\overline{U}'f(UwV)\overline{V}'$ is a normalized mapping. Thus we can rotate *z* by unitary transformations to ζz_0, where $\zeta \in C$, and

$$z_0 = \begin{pmatrix} z_{11}^0 & 0 & \dots & 0 & \dots & 0 \\ 0 & z_{22}^0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z_{mm}^0 & \dots & 0 \end{pmatrix}$$

Has $\|z_0\| = 1$ and $\zeta z_0 \in R_I$ or all $\zeta \in D$. Let $g(\zeta) = f(\zeta z_0) = \sum_{p=1}^{\infty} D_q^{jk}\zeta^p$ for $|\zeta| < 1$.
By definition,

$$\begin{aligned} |g(\zeta)|^2 &= \sum_{j=1}^{m}\sum_{k=1}^{n}\left|\sum_{p=1}^{\infty} D_p^{jk}\zeta^p\right|^2 \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n}\left(\sum_{p=1}^{\infty} D_p^{jk}\zeta^p\right)\left(\sum_{q=1}^{\infty}\overline{D_q^{jk}}\overline{\zeta}^q\right) \\ &\le \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{p,q=1}^{\infty}\left|D_p^{jk}\right|\left|D_q^{jk}\right||\zeta|^{p+q} \end{aligned}$$

By the triangle inequality.

$$\begin{aligned} &= \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{r=2}^{\infty}\sum_{p+q=r,\,p,q>0}\left|D_p^{jk}\right|\left|D_q^{jk}\right||\zeta|^r \\ &= \sum_{r=2}^{\infty}|\zeta|^r\sum_{p+q=r,\,p,q>0}\sum_{j=1}^{m}\sum_{k=1}^{n}\left|D_p^{jk}\right|\left|D_q^{jk}\right| \\ &\le \sum_{r=2}^{\infty}|\zeta|^2\sum_{p+q=r,\,p,q>0}\sum_{j=1}^{m}\sqrt{\sum_{k=1}^{n}\left|D_p^{jk}\right|^2}\sqrt{\sum_{k=1}^{n}\left|D_q^{jk}\right|^2} \end{aligned}$$

By Cauchy’s inequality

$$\leq \sum_{r=2}^{\infty} |\zeta|^r \sum_{p+q=r,\, p,q>0} m$$
$$= m \sum_{r=2}^{\infty} (r-1)|\zeta|^r$$
$$= m \frac{|\zeta|^2}{(1-|\zeta|)^2} .$$

For $\zeta z_0 = z$, $|\zeta| = \|z\|$, and $g(\zeta) = f(z)$. Then

$$|g(\zeta)| \leq \sqrt{m} \frac{\|z\|}{(1-\|z\|)} .$$

PROPOSITION 4.2. *Let f(z) be a convex mapping from* R_1 *into* C_{mxn} . *Let q be a positive integer. As in the proof of Lemma 4.1above, fix such a* z_0 *and let* $g(\zeta) = f(\zeta z_0) = \sum_{n=1}^{\infty} D_n^{jk} \zeta^n$ *for* $|\zeta| < 1$.
Then

$$\left|\frac{d^q g}{d\zeta^q}\right| \leq \sqrt{m} \frac{q!}{(1-\|z\|)^{q+1}}$$

Proof

$$\left|\frac{d^q g}{d\zeta^q}\right|^2 = \sum_{j=1}^{m}\sum_{k=1}^{n} \left|\sum_{p=q}^{\infty} p\ldots(p-q+1) D_p^{jk} \zeta^{p-q}\right|^2$$
$$= \sum_{j=1}^{m}\sum_{k=1}^{n} \left(\sum_{p=q}^{\infty} p\ldots(p-q+1) D_p^{jk} \zeta^{p-q}\right) \left(\sum_{r=1}^{\infty} r\ldots(r-q+1) \overline{D_r^{jk}} \overline{\zeta}^{p-q}\right)$$
$$\leq \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{p,r=q}^{\infty} p\ldots(p-q+1) r\ldots(r-q+1) \left|D_p^{jk}\right| \left|D_r^{jk}\right| |\zeta|^{p+r-2q}$$

By the triangle inequality

$$= \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{t=2q}^{\infty} \sum_{p+r=t,\, p,r\geq q} p\ldots(p-q+1) r\ldots(r-q+1) \cdot |\zeta|^{t-2q} \left|D_p^{jk}\right| \left|D_r^{jk}\right|$$
$$= \sum_{t=2q}^{\infty} |\zeta|^{t-2q} \sum_{p+r=t,\, p,r\geq q} p\ldots(p-q+1) r\ldots(r-q+1) \cdot \sum_{j=1}^{m}\sum_{k=1}^{n} \left|D_p^{jk}\right| \left|D_r^{jk}\right|$$
$$\leq \sum_{t=2q}^{\infty} |\zeta|^{t-2q} \sum_{p+r=t,\, p,r\geq q} p\ldots(p-q+1) r\ldots(r-q+1) \cdot \sum_{j=1}^{m} \sqrt{\sum_{k=1}^{n} \left|D_p^{jk}\right|^2} \sqrt{\sum_{k=1}^{n} \left|D_r^{jk}\right|^2}$$

By cauchy's inequality, and by Lemma 4.1, this is

$$\leq m \sum_{t=2q}^{\infty} |\zeta|^{t-2q} \sum_{p+r=t,\, p,r\geq q} p\ldots(p-q+1) r\ldots(r-q+1)$$
$$= m \sum_{t=2q}^{\infty} |\zeta|^{t-2q} \sum_{p=q}^{t-p} p\ldots(p-q+1)(t-q)\ldots(t-p-q+1)$$
$$= m \left(\frac{q!}{(1-|\zeta|)^{q+1}}\right)^2 .$$

For $|\zeta| = \|z\|$,

$$\left|\frac{d^q g}{d\zeta^q}\right| \leq \sqrt{m} \frac{q!}{(1-\|z\|)^{q+1}} .$$

Note that these results extend easily to the classical domains R_{II} and R_{III} .

Summary. Adhering to a standard method from one variable, we have infused some of the geometric theory of convex functions to convex mappings in several variables. For mappings of the ball in C^n and for mappings of the first classical domain onto convex sets, we have found frontiers on certain combinations of coefficients. Categorically the work carries over to R_{II} and R_{III}. These

estimates yield frontiers on the growth of the mappings and estimates on radial derivatives. All these estimates crusades to the best possible. The same coefficient estimate embraces estimates on other combinations of coefficients and on quantities such as the Jacobian. For these, the estimates are apparently not the best possible.

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