NORMALITY OF MEROMORPHIC FUNCTIONS BESIDES SHARING VALUES

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***ABSTRACT*: Certain gauges to determine the normality of group F of meromorphic functions in piece circle, which dividends values dependent on f ∈ F with their derivatives is acquired.**

***Keywords:* Meromorphic function; Normal family; Sharing values**

I. INTRODUCTION

Let f be meromorphic in a domain D ⊆ C: For a ∈ C, let

$$E_f(a) = \{z \in D: f(z) = a\} = f^{-1}(\{a\}) \cap D,$$

and let

$$E_f(\infty) = \{z \in D: z \text{ is a pole of } f\}.$$

For a ∈ C, the prolonged multifarious plane, dual functions f and g meromorphic in D remain said to distribute the value a if $E_f(a) = E_g(a)$.

If f and g are not a persistent meromorphic functions in C that share five values then f ≡ g.

The overhead is illustrious outcome of Nevanlinna [4]. Several authors established numerous associations amid f and g uncertainly share themselves littler than 5 values [5, 10, 11]. Correspondingly the association amid their progressions was acquired while they share values [2, 8] or small functions [7, 16]. Functions which share values with their derivatives and the association amid among them was also discussed [9]. Moreover, some differential polynomials of meromorphic functions which share values was considered. Initially, Schwick [14] is undoubtedly identified a link amid the normality criterion and shared values of meromorpohic functions. He substantiated the subsequent theorem:

THEOREM A Let F be a family of meromorphic functions in the unit disc Δ, and let a_1, a_2, a_3 be distinct complex numbers. If f and f_0 share a_1, a_2 and a_3 for every f ∈ F, then F is normal in Δ.

The concepts on shared values and normality was already utilized by numerous additional researchers [13]. Pang and Zalcman [12] substantiated the subsequent theorem:

THEOREM B Let F be a family of meromorphic functions on the unit disc Δ and let a and b be distinct complex numbers and c be a nonzero complex number. If for every f ∈ F,

$$E_f(0) = E_{f'}(a) \quad and \quad E_f(c) = E_{f'}(b),$$

where

$$E_f(a) = \{z \in \Delta: f(z) = a\},$$

then F is normal in Δ.

In Theorem B, the constants a, b, c are the same for every f ∈ F. We will demonstrate that the condition for the constants to be the same can be loose to some degree. Therefore we will consider constants a_f, b_f, c_f relying upon f. Obviously, these can't be taken discretionarily, and it is basic that the proportion $(a_f b_f / c_f^2)$ be steady and the focuses be consistently isolated. Indeedly, the accompanying theorem is elucidated

THEOREM 1 Let F be a family of meromorphic functions in the unit disc Δ. For each f ∈ F let a_f, b_f, c_f be distinct nonzero complex numbers such that $(a_f b_f / c_f^2) = M$ for some constant M. Let the spherical distance σ between the points a_f, b_f, c_f satisfy

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$$

for some m > 0: Let $E_f(0) = E_{f'}(a_f)$ and $E_f(c_f) = E_{f'}(b_f)$. Let M = (ab/c²), where a, b, c are distinct. If the elements of $E_f(c_f)$ and $E_f(0)$ are the only solutions of

$$f'(z) = \frac{a_f b}{a}\left(1 - \left(\frac{1}{c_f} - \frac{a}{c a_f}\right) f(z)\right)^2 \tag{1}$$

and

$$f'(z) = a_f\left(1 - \left(\frac{1}{c_f} - \frac{a}{ca_f}\right)f(z)\right)^2, \tag{2}$$

respectively. Then F is normal in Δ.

Note that the elements of $E_f(c_f)$ and $E_f(0)$ do satisfy (1) and (2), respectively.

Aimed at constraint that $a_f = 0$, b_f and c_f are distinct nonzero complex numbers depending on f, the subsequent dual theorems are substantiated,

THEOREM 2 Let F be a family of meromorphic functions in the unit disc Δ: For each f ∈ F suppose there exist nonzero complex numbers b_f, c_f satisfying
(i) b_f/c_f is a constant
(ii) $\min\{\sigma(0,b_f), \sigma(0,c_f), \sigma(b_f,c_f)\} \geq m$ for some $m > 0$
(iii) $E_f(0)=E_{f'}(0)$ and $E_f(c_f) = E_{f'}(b_f)$
Then F is normal in Δ.

THEOREM 3 Let F be a family of meromorphic functions in the unit disc Δ. For each f ∈ F suppose there exist nonzero complex numbers b_f, c_f satisfying
(i) b_f/c_f is a constant
(ii) $\min\{\sigma(0,b_f), \sigma(0,c_f), \sigma(b_f,c_f)\} \geq m$ for some $m > 0$
(iii) $E_f(0)=E_{f'}(b_f)$ and $E_f(c_f) = E_{f'}(c_f)$
Then F is normal in Δ.

II. PROOF OF THEOREMS

Proof of Theorem 1: For each f ∈ F, define a Mobius map g_f by

$$g_f(z) = \frac{z}{Az+B},$$

where

$$A = \frac{1}{c_f} - \frac{a}{ca_f} \quad \text{and} \quad B = \frac{a}{a_f}.$$

Then clearly

$$g_f^{-1}(z) = \frac{Bz}{1-Az} \quad \text{and} \quad (g_f^{-1})'(z) = \frac{B}{(1-Az)^2}$$

so that

$$g_f^{-1}(0) = 0, \quad g_f^{-1}(c_f) = c, \quad (g_f^{-1})'(0) = \frac{a}{a_f}, \quad (g_f^{-1})'(c_f) = \frac{b}{b_f}.$$

Now if z_1 is such that $F(z_1) = c_f$ then since $E_f(c_f)=E_{f'}(b_f)$ we have $f'(z_1)=b_f$ and so $(g_f^{-1} \circ f)(z_1) = g_f^{-1}(c_f) = c$ and $(g_f^{-1} \circ f)'(z_1) = (g_f^{-1})'(f(z_1))f'(z_1) = (g_f^{-1})'(c_f)b_f = b.$

We now show that $E_{g_f^{-1}\circ f}(c) = E_{(g_f^{-1}\circ f)'}(b)$. Let $z_o \in E_{g_f^{-1}\circ f}(c)$. Then $(g_f^{-1}\circ f)(z_o) = c = g_f^{-1}(c_f)$. Also g_f^{-1} being a Mobius map, is one-to-one so that $f(z_o) = c_f$ and so $f'(z_o) = b_f$. Thus $z_o \in E_{(g_f^{-1}\circ f)'}(b)$.

Next let $z_1 \in E_{(g_f^{-1}\circ f)'}(b)$. Then $(g_f^{-1})'(f(z_1))f'(z_1) = b$ and so

$$f'(z_1) = \frac{a_f b}{a}\left(1 - \left(\frac{1}{c_f} - \frac{a}{ca_f}\right)f(z_1)\right)^2. \tag{3}$$

Since only the elements of $E_f(c_f)$ satisfy (7), it follows that $z_1 \in E_f(c_f)$ and so $z_1 \in E_{g_f^{-1}\circ f}(c)$. Thus $E_{(g_f^{-1}\circ f)'}(b) \subset E_{g_f^{-1}\circ f}(c)$.

Next if $f(w_o) = 0$ then $f'(w_o) = a_f$ and so $(g_f^{-1}\circ f)(w_o) = 0$ and $(g_f^{-1}\circ f)'(w_o) = a$. Following a similar argument to the above we can show that $E_{g_f^{-1}\circ f}(0) = E_{(g_f^{-1}\circ f)'}(a)$.

Thus by Theorem B, the family $\mathcal{G} = \{(g_f^{-1} \circ f) : f \in \mathcal{F}\}$ is normal and hence equicontinuous in Δ. Therefore give $(\epsilon/k_m) > 0$, where k_m is the constant of Theorem E, there exist δ > 0 such that for the spherical distance σ(x, y) < δ,

$$\sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \frac{\epsilon}{k_m}$$

for each f ∈ F. Hence by Theorem E, for σ(x, y) < δ ,

$$\begin{aligned}\sigma(f(x), f(y)) &= \sigma((g_f \circ g_f^{-1} \circ f)(x), (g_f \circ g_f^{-1} \circ f)(y)) \\ &\leq k_m \sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) \\ &< \epsilon.\end{aligned}$$

Thus the family F is equicontinuous in Δ. This completes the proof.

Proof of Theorems 2 and 3 For each $f \in \mathcal{F}$, define a Mobius map g_f by $g_f(z) = (c_f/c)z$. The condition that $E_f(c_f) = E_{f'}(b_f)$, g_f^{-1} is one-to-one and $g_f^{-1}(c_f) = c$ ensure that $E_{g_f^{-1} \circ f}(c) = E_{(g_f^{-1} \circ f)'}(b)$.

Also $E_{g_f^{-1} \circ f}(0) = E_{(g_f^{-1} \circ f)'}(0)$ is a consequence of $E_f(0) = E_{f'}(0)$.

Applying Theorem B with $a = 0$, the family $\{(g_f^{-1} \circ f) | f \in \mathcal{F}\}$ is normal in Δ. As in Theorem 1, it now follows that the family $\mathcal{F}$ is normal in Δ.

For the proof of Theorem 3, we utilize $E_f(0) = E_{f'}(b_f)$ and $E_f(c_f) = E_{f'}(c_f)$ to show that $E_{g_f^{-1} \circ f}(0) = E_{(g_f^{-1} \circ f)'}(b)$ and $E_{g_f^{-1} \circ f}(c) = E_{(g_f^{-1} \circ f)'}(c)$ and apply Theorem B with a non-zero $b = c$ distinct from a with c nonzero. Theorem 3 now follows as above.

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